Negation from the perspective of neighborhood semantics

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Abstract

This paper explores many important properties of negations using the neighborhood semantics. We generalize the correspondence between the properties of negations and the conditions on the frames and also establish the duality between distributive lattices with negation and descriptive general negation-neighborhood frames.

Key words: negation, neighborhood frames, distributive logic with negation, correspondence, duality

1 Introduction

Weaker negations than classical one have been discussed by many papers, like, [6], [7], [9], [10], [18]. In [6] the author discusses negation on the base of language L including $\rightarrow, \wedge, \vee$, and $\neg$. The weakest system in [6] is the negationless fragment of Heyting propositional calculus H with contraposition i.e. if $A \rightarrow B$ then $\neg B \rightarrow \neg A$ and one of De Morgan laws: $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$. The author uses N frame $Fr = \langle X, R_I, R_N \rangle$. R_I and R_N satisfy some conditions. R_I and R_N deal with $\rightarrow$; R_N deals with $\neg$. [6] proves that N is complete with respect to the class of N frames. Extending on N the author furthermore explores more relations between properties on negations and the conditions on R_I and R_N, like $A \rightarrow \neg\neg A$, $A \wedge \neg A \rightarrow B$ etc. [9] investigates the relation between the two semantics star and perp (See 6.2 below). The

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semantic structure in [9] is partially ordered set $\langle P, \leqslant, \neg\rangle$. Started from subminimal negation, i.e. if $a \leqslant b$ then $\neg b \leqslant \neg a$, the author studies the following five kinds of negations: (1) Galois connected negations, i.e. two subminimal negations $\neg$ and $-$ with Galois property: given two partially ordered sets $\langle P, \leqslant\rangle$, $\langle P', \leqslant'\rangle$, $\neg : P \to P'$ and $- : P' \to P$ then $a \leqslant -b$ iff $b \leqslant' \neg a$ for all $a \in P$ and $b \in P'$; (2) minimal negation, i.e. $\neg = -$ and $a \leqslant \neg\neg a$; (3) De Morgan negation, i.e. minimal negation satisfying $\neg\neg a \leqslant a$; (4) intuitionistic negation, i.e. minimal negation satisfying if $a \leqslant b$ and $a \leqslant \neg b$ then $a = 0$; (5) ortho negation, which is both De Morgan negation and the negation of intuitionistic logic. The author explores conditions on perp and star corresponding to the above six kinds of negation and the relation between the two treatments of negation. The work in [9] is based on partially ordered sets. [9] doesn't investigate the correspondence between $\leqslant$ and De Morgan laws based on subminimal negation. Both [16] and [18] investigate the frame semantics $\langle P, C, \sqsubseteq\rangle$. In both papers the weakest logic the semantics characterizes corresponds to S with N1 and N3 in our paper. Further conditions on C in the frame correspond to more other properties of negation. [14] and [1] are all based on distributive modal algebras ($DMAs$), which includes two weaker negation operators $\lhd$ and $\rhd$ besides modal operators box, diamond and Boolean connectives conjunction, disjunction, false and truth. $DMAs$ can represent various forms of weak negations although they don't include negations directly. For example, the operator $\lhd$ turns conjunction into disjunction and true into false, while $\rhd$ turns disjunction into conjunction and false into true. Thus in a modal algebra $\lhd$ behaves as $\Diamond\neg$ while $\rhd$ behaves as $\neg\Diamond$. In the two papers weaker negations appears at the same time and are all related with modal operators. Now the following questions seem to be nature: How does the frame look if the negation is only antitone based on the distributive lattice? What is the situation when we add more properties to it one by one? The paper answers all these questions.

In the paper we follow the lines the way of [14] and [1]. The language used here is similar to the one in the two papers. But we focus only on negation. We think that among all properties of negation antitony is the most basic and fundamental. So we first study the weakest S, i.e. distributive logic with a negation being only antitone. Negation-neighborhood frames are used to capture the property of negation. We prove S's completeness with respect to the class of negation-neighborhood frames by the representation theorem. The readers unfamiliar with representation theorems are referred to [4], Chapter 4 in [3], Chapter 5 in [14]. Then we study other properties of negation one by one from two angles: correspondence and canonicity. We refer those readers unfamiliar with the concepts of correspondence and canonicity to Chapter 5 in [14], [17] or Section 5 below. After this we compare our results with those in the literature and discuss their relations, which demonstrates that the way here is more general in the sense that it can capture some additional single property of the negation with antitony. The last part of the paper is devoted to the duality between the category of distributive lattices with negation and the category of descriptive general negation-neighborhood frames. We discuss mainly topological duality, which generalizes the one in [7].

2 Syntax

We will be working with the following language $\mathcal{L}$. $\mathcal{L}$ contains $\vee, \wedge, \bot, \top, \neg$, where $\vee$ and $\wedge$ are binary, $\bot$ and $\top$are nullary, $\neg$ is unary. And we fix a set $\Phi = \{x_1, x_2, \ldots\}$. Then we can form the formulas using the above connectives. The set of all formulas is denoted by $Form(\Phi)$. But in order to talk about the logics we need the concept of sequent. A *sequent* is simply a pair of formulas of the form (α, β), which will be written $\alpha \vdash \beta$.

Definition 1. *A distributive logic with negation is a set* Λ *of sequents such that* Λ *contains the following sequents and inference rules:*
Sequents (*Axioms*)
$x \vdash x$
$\bot \vdash x \quad x \vdash \top$
$x \wedge (y \vee z) \vdash (x \wedge y) \vee (x \wedge z)$
$x \vdash x \vee y \quad y \vdash x \vee y \quad x \wedge y \vdash x \quad x \wedge y \vdash y$
Inference rules
If $\alpha \vdash \beta$ *and* $\beta \vdash \gamma$ *then* $\alpha \vdash \gamma$ *(cut)*
If $\alpha \vdash \beta$ *then* $\alpha(\gamma/x) \vdash \beta(\gamma/x)$ *(substitution)*
If $\alpha \vdash \gamma$ *and* $\beta \vdash \gamma$ *then* $\alpha \vee \beta \vdash \gamma$
If $\alpha \vdash \beta$ *and* $\alpha \vdash \gamma$ *then* $\alpha \vdash \beta \wedge \gamma$
If $\alpha \vdash \beta$ *then* $\neg\beta \vdash \neg\alpha$

It is easy to see that the family of distributive logics with negation is closed under intersection. So there exists a smallest distributive logic with negation, which is denoted by S. If $\alpha \vdash \beta$ and $\beta \vdash \alpha$, we denote this by

$\alpha \dashv\vdash \beta$. It is easy to check that $\dashv\vdash$ is a equivalent relation on $Form(\Phi)$. If Γ is a set of sequents, then $S.\Gamma$ denotes the smallest distributive logic with negation containing Γ.

3 Semantics

3.1 Neighborhood semantics

Let F is a partially ordered set. $\mathcal{U}(F) = \{U \subseteq F \mid s \in U \ \& \ s \leqslant t \Rightarrow t \in U\}$.

Definition 2. *A negation-neighborhood frame (for short frame or NF) $\mathbb{F}$ is a triple $\langle F, \leqslant, N\rangle$, where F is a non-empty set, $\leqslant$ a partial order on F and N: $F \to \mathcal{P}(\mathcal{U}(F))$ satisfying the following conditions:*
(1) *for any s_1, $s_2 \in F$, if $s_1 \leqslant s_2$, then $N(s_1) \subseteq N(s_2)$*
(2) *for any $s \in S$, and $X, Y \in \mathcal{U}(F)$, if $X \subseteq Y$ and $Y \in N(s)$, then $X \in N(s)$.*

A valuation on a NF is a map $V : \Phi \to \mathcal{U}(F)$.

A model is a pair $(\mathbb{F}, V)$ consisting of a frame $\mathbb{F}$ and a valuation V on $\mathbb{F}$.

Remarks: N in the frame is called the neighborhood function. The value of N at s is called the collection of the neighborhood of s. We use the upsets in the above definition. We call it Up-$frames(models)$. In fact we can replace $\mathcal{U}(F)$ with the collections of downsets $\mathcal{D}(F)$ and replace $N(s_1) \subseteq \mathrm{N}(s_2)$ with $N(s_2) \subseteq \mathrm{N}(s_1)$ then we get another version of frame. we call them $Down$-$frames(models)$, which are equivalent to the Up-frames(models).

Definition 3. *Given a model $\mathbb{M} = \langle \mathbb{F}, V\rangle$ the truth relation $\Vdash$ between points and formulas is defined by the following induction:*
(1) *For $x \in \Phi$ we define $\mathbb{M}, s \Vdash x$ if and only if $s \in V(x)$;*
(2) *For any α, β, we put*
(a) $\mathbb{M}, s \Vdash \alpha \vee \beta$ *if and only if* $\mathbb{M}, s \Vdash \alpha$ *or* $\mathbb{M}, s \Vdash \beta$;
(b) $\mathbb{M}, s \Vdash \alpha \wedge \beta$ *if and only if* $\mathbb{M}, s \Vdash \alpha$ *and* $\mathbb{M}, s \Vdash \beta$;
(c) $\mathbb{M}, s \Vdash \neg\alpha$ *if and only if* $[\![\alpha]\!] \in N(s)$, *where* $[\![\alpha]\!] := \{s \mid s \Vdash \alpha\}$;
(d) $\mathbb{M}, s \nVdash \bot$;
(e) $\mathbb{M}, s \Vdash \top$.

Intuitively the condition c says that whether the negation of a proposition α holds at the state s depends on the extension of α is in the neighborhood of s or not. Or we can also say that the neighborhood of a state determines which propositions of the form $\neg a$ hold at the state. When we put some condition on the neighborhoods on the frame we get some corresponding property on $\neg$. In the definition of a frame above we require that N is downwards closed. This requirement leads to the fact that $\neg$ is antitone in the logic. Based on this we can put more conditions on N, then $\neg$ has more other properties, i.e. it will become a stronger negation. On the contrary $\neg$ will behave more like a modality if we require that N is upwards closed . See [5] and [15].

Definition 4. *A model $\mathbb{M} = \langle \mathbb{F}, V\rangle$ satisfies a sequent $\alpha \vdash \beta$,written $\mathbb{M} \Vdash \alpha \vdash \beta$, if for each $s \in F$ with $s \Vdash \alpha$ we have $s \Vdash \beta$. A frame $\mathbb{F}$ validates a sequent $\alpha \vdash \beta$, written $\mathbb{F} \Vdash \alpha \vdash \beta$, if each model $(\mathbb{F}, V)$ satisfies $\alpha \vdash \beta$. A frame $\mathbb{F}$ validates a set of sequents Γ, if $\mathbb{F} \Vdash \alpha \vdash \beta$ for each sequent $\alpha \vdash \beta \in \Gamma$.*

3.2 Algebraic semantics

If we view $\alpha \vdash \beta$ as an algebraic inequality $\alpha \preccurlyeq \beta$, which is equivalent to $\alpha \wedge \beta \approx \alpha$, we get the distributive lattice with negation (DLN). The observation leads the following definition:

Definition 5. *A distributive lattice with negation (DLN) is a lattice $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg\rangle$, where $\langle A, \vee, \wedge, \bot, \top, \rangle$ is a bounded distributive lattice, and $\neg$ is a unary operator satisfying the following condition: for all a, $b \in A$, if $a \leqslant b$, then $\neg a \leqslant \neg b$.*

If an DLN $\mathbb{A}$ validates an algebraic inequality $\alpha \preccurlyeq \beta$, we denote it by $\mathbb{A} \models \alpha \preccurlyeq \beta$. When we say that DLN $\mathbb{A}$ validates a sequent we mean that it validates its corresponding algebraic inequality.

The following proposition is straightforward :

Proposition 6. *$\alpha \vdash \beta \in S$ iff $\mathbb{A} \models \alpha \preccurlyeq \beta$ for any DLN $\mathbb{A}$.*

4 Completeness

With the help of DLN we can prove the completeness theorem of S with respect to the negation-neighborhood semantics by the representation theorem.

Definition 7. *Let $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg \rangle$ be a DLN. The prime filter frame $\mathbb{A}_\bullet$ of $\mathbb{A}$ is $\langle Pf\mathbb{A}, \subseteq, N_\neg \rangle$, where $Pf\mathbb{A}$ is the collection of prime filters of $\mathbb{A}$, $N_\neg$ is a map from $Pf\mathbb{A}$ to $\mathcal{P}(\mathcal{U}(Pf\mathbb{A}))$ and satisfies the following conditions:*

- *for clopen upset $\widehat{b}$ of $Pf\mathbb{A}$, $\widehat{b} \in N_\neg(u)$ iff $\neg b \in u$.*

- *for open upset $\mathcal{O}$ of $Pf\mathbb{A}$, $\mathcal{O} \in N_\neg(u)$ iff for any clopen upset $\widehat{b}$, if $\widehat{b} \subseteq \mathcal{O}$, then $\widehat{b} \in N_\neg(u)$.*

- *for any upset X of $Pf\mathbb{A}$, $X \in N_\neg(u)$ iff there is an open upset $\mathcal{O}$ with $\mathcal{O} \supseteq X$ and $\mathcal{O} \in N_\neg(u)$.*

Explanations: The dual space of the underlying distributive lattice $\langle A, \vee, \wedge, \bot, \top \rangle$ of $\mathbb{A}$ is called Priestly space or ordered Stone space, comprising $\langle Pf\mathbb{A}, \mathcal{T} \rangle$. Sets of the form $\widehat{b} = \{u \mid b \in u\}$ for $b \in A$ are all clopens up-sets in the topology. They and their complements form a subbasis of the topology. See [4] for more details about lattice and its duality.

Proposition 8. *Let $\mathbb{A}$ be an DLN. Then $\mathbb{A}_\bullet$ is a negation-neighborhood frame.*

Proof. Let $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg \rangle$. Then $\mathbb{A}_\bullet = \langle Pf\mathbb{A}, \subseteq, N_\neg \rangle$. Assume that $u, v \in Pf\mathbb{A}, u \subseteq v$ and $X \in N_\neg(u)$. By Definition 7 $X \in N_\neg(u)$ implies that there is an open upset $\mathcal{O} \supseteq X$ and $\mathcal{O} \in N_\neg(u)$. That means there is an open $\mathcal{O} \supseteq X$, for any $\widehat{b}$, if $\widehat{b} \subseteq \mathcal{O}$, then $\neg b \in u$. Since $u \subseteq v$ we have immediately that there is an $\mathcal{O} \supseteq X$, for any $\widehat{b}$, if $\widehat{b} \subseteq \mathcal{O}$, then $\neg b \in v$. By Definition 7 again we get $X \in N_\neg(v)$.

Now we assume that $X \subseteq Y$ and $Y \in N_\neg(u)$. $Y \in N_\neg(u)$ implies that there is an open upset $\mathcal{O} \supseteq Y$ and $\mathcal{O} \in N_\neg(u)$. Since $X \subseteq Y$ we have that there is an open upset $\mathcal{O} \supseteq X$ and $\mathcal{O} \in N_\neg(u)$. So $X \in N_\neg(u)$. □

Definition 9. *Let* $\mathbb{F} = \langle F, \leqslant, N\rangle$ *be a negation-neighborhood frame. The complex lattice* F^+ *of* $\mathbb{F}$ *is defined as* $\langle \mathcal{U}(\mathbb{F}), \cup, \cap, \emptyset, F, \neg_N\rangle$*, where* $\neg_N : \mathcal{U}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ *and satisfies* $\neg_N(U) = \{s \in F \mid U \in N(s)\}$*.*

Proposition 10. *The complex lattices of negation-neighborhood frames are DLNs.*

Proof. Let $\mathbb{F} = \langle F, \leqslant, N\rangle$ be a negation-neighborhood frame.Then $\mathbb{F}^+ = \langle \mathcal{U}(\mathbb{F}), \cup, \cap, \emptyset, F, \neg_N\rangle$.
First we should show that for any $U \in \mathcal{U}(\mathbb{F})$, $\neg_N(U)$ is indeed an upset of F. Assume that $s \leqslant t$ and $s \in \neg_N(U)$. By Definition 2 $s \leqslant t$ implies $N(s) \subseteq N(t)$. And by Definition 9 $s \in \neg_N(U)$ implies $U \in N(s)$. Hence $U \in N(t)$, which implies $t \in \neg_N(U)$. So $\neg_N$ is an upset of F.
Now we assume that $U \subseteq V$ and $s \in \neg_N(V)$. So $V \in N(s)$. By the property of N we have $U \in N(s)$. So $s \in \neg_N(U)$. Hence $\neg_N(V) \subseteq \neg_N(U)$. Therefore $\neg_N$ is antitone. □

Proposition 11. *For any sequent* $\alpha \vdash \beta$*,* $\mathbb{F} \Vdash \alpha \vdash \beta$ *iff* $\mathbb{F}^+ \models \alpha \preccurlyeq \beta$*.*

Proof. This follows from the observation that a valuation is actually an assignment. □

Proposition 12. *Let* $\mathbb{A}$ *be a DLN.* $\mathbb{A}$ *is embeddable into* $(\mathbb{A}_\bullet)^+$*.*

Proof. Let $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg\rangle$. Then $(\mathbb{A}_\bullet)^+ = \langle \mathcal{U}(Pf\mathbb{A}), \cup, \cap, \emptyset, Pf\mathbb{A}, \neg_{N_\neg}\rangle$. Let $f : a \mapsto \widehat{a}$. The proof that f is an embedding from $\langle A, \vee, \wedge, \bot, \top\rangle$ to $\langle \mathcal{U}(Pf\mathbb{A}), \cup, \cap, \emptyset, Pf\mathbb{A}\rangle$ is standard. See [4]. Now we show that $f(\neg a) = \neg_{N_\neg}(fa)$, which follows from the following sequence of equations: $\neg_{N_\neg}(fa) = \neg_{N_\neg}(\widehat{a}) = \{u \in Pf\mathbb{A} \mid \widehat{a} \in N_\neg(u)\} = \{u \in Pf\mathbb{A} \mid \neg a \in u\} = \widehat{\neg a} = f(\neg a)$ □

Explanations: The readers familiar with the canonical extensions of distributive lattice recognize that $(\mathbb{A}_\bullet)^+$ is actually the canonical extension of $\mathbb{A}$, i.e. $(\mathbb{A}_\bullet)^+ = \mathbb{A}^\pi$. $\neg_{N_\neg}$ is actually the canonical extension of $\neg$, i.e. $\neg^\pi$. The canonical extension f^π of a homomorphism f between two bounded distributive lattices maps closed sets and open sets to closed sets and open sets respectively. Besides f^π there is another canonical extension f^σ which uses closed sets. If f is order-preserving or turns meet to join or turns join to meet, then $f^\pi = f^\sigma$. In such cases we can equivalently use closed subsets to define $N_\neg$ and then $\neg_{N_\neg}$ as follows:

- for closed upset $\mathcal{C}$ of $Pf\mathbb{A}$, $\mathcal{C} \in N_\neg(u)$ iff there is a clopen upset $\widehat{b}$ with $\widehat{b} \supseteq \mathcal{C}$ and $\widehat{b} \in N_\neg(u)$.

- for any upset X of $Pf\mathbb{A}$, $X \in N_\neg(u)$ iff for any closed upset $\mathcal{O}$ if $\mathcal{C} \subseteq X$ then $\mathcal{C} \in N_\neg(u)$.

For more details about the canonical extension see [12].

Theorem 13. *S is complete with respect to the class of the negation-neighborhood frames.*

Proof. Suppose $\alpha \vdash \beta \notin S$. We will use a special algebra: Lindenbaum-Tarski algebra $\mathfrak{L}_S$. $\mathfrak{L}_S = \langle Form(\Phi)/_{\dashv\vdash}, \vee, \wedge, [\bot], [\top] \rangle$, where $Form(\Phi)/_{\dashv\vdash}$ is the equivalence class of $Form(\Phi)$ under the relation $\dashv\vdash$. It is easy to see that $\mathfrak{L}_S$ is a DLN. Actually it is the free algebra over Φ in the variety corresponding to S. It is not difficult to verify the following fact:
For any sequent $\alpha \vdash \beta$, $\alpha \vdash \beta \in S$ iff $\mathfrak{L}_S \models \alpha \vdash \beta$.
So we can infer as follows: $\alpha \vdash \beta \notin S$
$\Rightarrow \mathfrak{L}_S \not\models \alpha \preccurlyeq \beta$ by the above fact
$\Rightarrow ((\mathfrak{L}_S)_\bullet)^+ \not\models \alpha \preccurlyeq \beta$ $\mathfrak{L}_S$ is the subalgebra of $((\mathfrak{L}_S)_\bullet)^+$
$\Rightarrow (\mathfrak{L}_S)_\bullet \not\Vdash \alpha \vdash \beta$ by Proposition 11

□

5 Further correspondence

In this section we investigate further seven sequents involving negation. We study the canonicity of sequents and the correspondences between sequents and the properties on frames. We call a sequent canonical if the canonical frame of any logic containing the sequents validates the sequent. We call a sequent corresponding to the property if the sequent is valid on any class of frames with a property and any frame validating the sequent has the property. We begin from the simplest: the relations between $\top, \bot, \neg\top, \neg\bot$. By the property of $\top$ and $\bot$ the two sequents $\neg\bot \vdash \top$ and $\bot \vdash \neg\top$ hold in S. But the other two don't generally.

N1 $\top \vdash \neg\bot$

Obviously it corresponds to the condition: for any $s \in F, \emptyset \in N(s)$. Is it canonical? The answer is Yes.

Proposition 14. $\top \vdash \neg\bot$ *corresponds to the condition: for any* $s \in F, \emptyset \in N(s)$. *And it is canonical.*

Proof. Take any u in $(\mathfrak{L}_{S.N1})_\bullet$. Since $\top \in u$, so $\neg\bot \in u$ by $\top \vdash \neg\bot$. Furthermore $\emptyset$ should be in $N(u)$ since $\emptyset = \widehat{\bot}$ by Definition 7. □

N2 $\neg\top \vdash \bot$

Proposition 15. $\neg\top \vdash \bot$ *corresponds to the condition: for any* $s \in F, F \notin N(s)$. *And it is canonical.*

Proof. It is easy to see the correspondence. Now we take any u in $(\mathfrak{L}_{S.N2})_\bullet$. Suppose $Pf(F) \in N(u)$. Since $Pf(F) = \widehat{\top}$, that means $\neg\top \in u$, then $\bot \in u$ by $\neg\top \vdash \bot$, which contradicts the fact that u is a filter. So $\neg\top \vdash \bot$ is canonical. □

Now we begin to explore the famous De Morgan Laws. By the antitony of $\neg$ half of them are theorems in S, i.e. $\neg(x \vee y) \vdash \neg x \wedge \neg y$ and $\neg x \vee \neg y \vdash \neg(x \wedge y)$. But the other two do not usually hold.

N3 $\neg x \wedge \neg y \vdash \neg(x \vee y)$

Obviously it corresponds to the second-order condition on the frame: for any subsets X and Y of F, any point s in F if $X \in N(s)$ and $Y \in N(s)$, then $X \cup Y \in N(s)$. We can do a better job than this: to reduce N to a binary relation on F under certain condition! Recall that $\triangleright$ in [17] satisfies N1 and N3. $\triangleright$ can be explained by a binary relation on F. Now we add only N3 to S. Generally it is impossible to deal with $\triangleright$ by a relation on F. However under some condition we can do so. This is because we can prove that N3 and some property has so-called canonical pseudocorrespondence which helps us succeed reducing N to a binary relation. We call a sequent and a property pseudocorrespondence if the canonical frame of any logic containing the sequent has the property and the complex lattice of any frame with the property validates the sequent. Common correspondences between sequents and a property are about any frames, but pseudocorrespondences are only about canonical frames. For convenient to formulate some results we first define R_1, the reduction of N as $R_1(s) = \bigcup_{X \in N(s)} X$. we claim that N3 and $R_1(s) \in N(s)$ is canonical pseudocorrespondent. But first we need do some preliminary jobs.

Lemma 16. [1] *In a compact totally order-disconnected space for any clopen upset a, open upsets o_1, o_2 if $a \subseteq o_1 \cup o_2$ then there exist a_1 and a_2 s.t. $a_1 \subseteq o_1, a_2 \subseteq o_2$ and $a = a_1 \cup a_2$.*

Proof. At first we state three facts without proof since they are well-know in topology.

- Fact 1 In a compact totally order-disconnected space $\mathbb{X} = (X, \tau)$, if $x \not\geq y$ then there is a clopen upset a s.t. $x \notin a$ and $y \in a$.

Fact 1 is very well-known. By Fact 1, it is not difficult infer the following Fact 2:

- Fact 2 Let c is a closed upset in a compact totally order-disconnected space. If $x \notin c$, then there is a clopen upset a s.t. $x \notin a$ and $c \subseteq a$.

Similarly Using Fact 2, we can easily prove the following Fact 3:

- Fact 3 Let c_1 and c_2 are closed upsets in a compact totally order-disconnected space. If $c_1 \cap c_2 = \emptyset$ then there is a clopen upset a s.t. $c_1 \cap a = \emptyset$ and $c_2 \subseteq a$.

Now we can dive into proving the lemma. Let $c_1 = a \cap \bar{o}_2$ and $c_2 = a \cap \bar{o}_1$. Then $c_1 \cap c_2 = \emptyset$. So by Fact 3, there is clopen b s.t. $c_1 \subseteq b$ and $c_2 \cap b = \emptyset$. Now take $a_1 = a \cap b$ and $a_2 = a \cap \bar{b}$. It is easy to see $a = a_1 \cup a_2$.
$c_2 \cap b = \emptyset$ implies $(a \cap \bar{o}_1) \cap b = \emptyset$. $a_1 \cap (o_2/o_1) = \emptyset$ since $a_1 \cap (o_2/o_1) \subseteq (a \cap \bar{o}_1) \cap b$. Then $a_1 \subseteq o_1$ since $a_1 \subseteq a \subseteq o_1 \cup o_2$.
$c_1 \subseteq b$ implies $\bar{b} \cap c_1 = \emptyset$, i.e. $\bar{b} \cap (a \cap \bar{o}_2) = \emptyset$. And $a_2 \cap (o_1/o_2) \subseteq \bar{b} \cap (a \cap \bar{o}_2)$. Hence $a_2 \cap (o_1/o_2) = \emptyset$. Therefore $a_2 \subseteq o_2$. □

Corollary 17. *In a compact totally order-disconnected space if clopen $a \subseteq \bigcup o_i$, where o_i is open for each i, then there are finite clopens $a_1, \cdots, a_n$, s.t. for each i with $1 \leq i \leq n$, $a_i \subseteq o_i$ and $a = \bigcup_{1 \leq i \leq n} a_i$.*

Proposition 18. *$\neg x \wedge \neg y \vdash \neg(x \vee y)$ and $R_1(s) \in N(s)$ are canonical pseudocorrespondent.*

[1] Dr. Yde venema informed me of the lemma.

Proof. It is obvious that for any frame $\mathbb{F}$, if it satisfies $R_1(s) \in N(s)$, then its complex lattice validates $\neg x \wedge \neg y \vdash \neg(x \vee y)$. We now prove that for any distributive logic with negation Λ if $\neg x \wedge \neg y \vdash \neg(x \vee y) \in \Lambda$ then $R_1(s) \in N(s)$ in $(\mathfrak{L}_\Lambda)_\bullet$.

For each $X_i \in N(u)$, there is an open subset $\mathcal{O}_i \supseteq X_i$ and for any clopen $\widehat{a} \subseteq \mathcal{O}_i$, $\neg a \in u$. So $\bigcup \mathcal{O}_i \supseteq \bigcup X_i$. Now consider any $\widehat{a} \subseteq \bigcup \mathcal{O}_i$. $\widehat{a}$ is compact since it is closed. Since $\bigcup \mathcal{O}_i$ is open there are finite $\mathcal{O}_1, \cdots, \mathcal{O}_n$ s.t. $\widehat{a} \subseteq \bigcup_{1 \leq i \leq n} \mathcal{O}_i$. By Corollary 17 there are finite $\widehat{a_1}, \cdots, \widehat{a_n}$, s.t. for each i with $1 \leq i \leq n$, $\widehat{a_i} \subseteq \mathcal{O}_i$ and $\widehat{a} = \bigcup_{1 \leq i \leq n} \widehat{a_i}$. So for each i with $1 \leq i \leq n$, $\neg a_i \in u$. Furthermore $\neg a_1 \wedge \cdots \wedge \neg a_n \in u$ which infers $\neg(a_1 \vee \cdots \vee a_n) \in u$ by $\neg x \wedge \neg y \vdash \neg(x \vee y)$. Hence $\neg a \in u$ since $\widehat{a} = \bigcup_{1 \leq i \leq n} \widehat{a_i}$. Then $\widehat{a} \in N(u)$, which implies $\bigcup_{1 \leq i \leq n} \mathcal{O}_i \in N(u)$. Therefore $R_1(u) \in N(u)$. □

Notably that N is empty does not mean that $R_1(s)$ is empty but means that $R_1(s)$ does not exist. So in order to get the binary relation we should require that $N(s)$ is not empty for each $s \in F$. Equivalently this is to say $\emptyset \in N(s)$ for each $s \in F$ since N is downward closed.

But we can change slightly the definitions of frame and satisfaction so as to get the perfect match between the syntax and the semantics.

Definition 19. *An R_1 frame is a triple $\langle F, \leqslant, K, R_1 \rangle$, where F and $\leqslant$ are the same as before, $K = \{s \in F : s \nVdash \neg\alpha$ for any $\alpha\}$ and R_1 is a binary relation on F satisfying $R_1(s) \subseteq R_1(t)$ whenever $s \leqslant t$ for any $s, t \in F$. The concepts of models and satisfactions are as before except replacing (c) in the Definition 2 with the following (c_1):*

(c_1) *$\mathbb{M}, s \Vdash \neg\alpha$ if and only if $s \notin K$ and for any $t \in F$ if $t \Vdash \alpha$ then $t \in R_1(s)$.*

Proposition 20. *Suppose $R_1(s) \in N(s)$. $s \Vdash \neg\alpha$ iff for any $t \in F$ if $t \Vdash \alpha$ then $t \in R_1(s)$.*

Proof. By Definition 3 it easy to see that $s \Vdash \neg\alpha$ implies $[\![\alpha]\!] \subseteq R_1(s)$. The other direction follows from the fact that N is downward closed. □

Finally we achieve the desired result:

Proposition 21. *Any S's extension with $\neg x \wedge \neg y \vdash \neg(x \vee y)$ is complete with respect to the class of R_1 frames.*

N4 $\neg(x \wedge y) \vdash \neg x \vee \neg y$

It is obvious that it corresponds to the second order condition: for any subsets X and Y of F, if $X \cap Y \in N(s)$, then $X \in N(s)$ or $Y \in N(s)$. But again we can have a better result under some condition.
Let $R_2(s) = \bigcap_{X \notin N(s)} X$. We have the following proposition:

Proposition 22. *$\neg(x \wedge y) \vdash \neg x \vee \neg y$ and $R_2(s) \notin N(u)$ are canonical pseudocorrespondent.*

Proof. It is obvious that for any frame $\mathbb{F}$, if it satisfies $R_2(s) \notin N(s)$, then its complex lattice validates $\neg(x \wedge y) \vdash \neg x \vee \neg y$. We now prove that for any distributive logic with negation Λ if $\neg(x \wedge y) \vdash \neg x \vee \neg y$ then $R_2(s) \notin N(u)$ in $(\mathcal{L}_\Lambda)_\bullet$. $X \in N(u)$ iff for any closed $\mathcal{C} \subseteq X$, there is a $\widehat{a}$ s.t $\widehat{a} \supseteq \mathcal{C}$ and $\neg a \in u$. (Here we use f^σ. See the remark below Definition 7.) Hence for each $X_i, X_i \notin N(u)$ iff there is a $\mathcal{C}_i \subseteq X_i$ s.t. for any $\widehat{a}$, if $\widehat{a} \supseteq \mathcal{C}_i$ then $\neg a \notin u$. Now we take $\bigcap \mathcal{C}_i$. Obviously $\bigcap \mathcal{C}_i \subseteq \bigcap X_i$. Take any $\widehat{a}$ with $\widehat{a} \supseteq \bigcap \mathcal{C}_i$. By Lemma 16 it is not hard to see that there are finite clopens $\widehat{a_1}, \cdots, \widehat{a_n}$ s.t. for each $i, \widehat{a_i} \supseteq \mathcal{C}_i$ and $\widehat{a} = \widehat{a_1} \cap \cdots \cap \widehat{a_n} = \widehat{a_1 \wedge \cdots \wedge a_n}$. Since for each i, $\neg a_i \notin u$, $\neg a_i \vee \cdots \vee \neg a_n \notin u$ by the fact that u is prime. Hence $\neg(a_i \wedge \cdots \wedge a_n) \notin u$ by the axiom $\neg(x \wedge y) \vdash \neg x \vee \neg y$, i.e. $\neg a \notin u$. Therefore $R_2(s) \notin N(u)$. $\square$

Similar to the case of R_1 there is no subset of F not in $N(s)$ if $N(s)$ is full for some point s. In this case we can not reduce the complement of $N(s)$ to R_2. Therefore, in order to get R_2, we should require that $N(s)$ is not full for each s in F. Equivalently that means $F \notin N(s)$ because N is downward closed. Analogue to above discuss about N3, the above observation leads to the following definition:

Definition 23. *An R_2 frame is a triple $\langle F, \leqslant, K, R_2 \rangle$, where F and $\leqslant$ are the same as before, $K = \{s \in F : s \Vdash \neg\alpha$ for any $\alpha\}$ and R_2 is a binary relation on F satisfying $R_2(s) \subseteq R_2(t)$ whenever $s \leqslant t$ for any $s, t \in F$. The concepts of models and satisfactions are as before except replacing (c) in the Definition 2 with the following (c_2):*

(c_2) $\mathbb{M}, s \Vdash \neg\alpha$ *if and only if* $s \in K$ *or there is some* $t \in R_2(s)$ *s.t.* $t \nVdash \alpha$.

Proposition 24. *Suppose $R_2(s) \notin N(s)$. $s \Vdash \neg\alpha$ iff there exists a t such that $t \in R_2(s)$ and $t \nVdash \alpha$.*

Proof. It is similar to Proposition 20. □

Finally we obtain the result we desire:

Proposition 25. *Any S's extension with $\neg(x \wedge y) \vdash \neg x \vee \neg y$ is complete with respect to the class of R_2 frames.*

N5 $x \wedge \neg x \vdash \bot$

Proposition 26. *$x \wedge \neg x \vdash \bot$ corresponds to the condition: if $s \in X$, then $X \notin N(s)$. And it is canonical.*

Proof. Correspondence is obvious. We just show the canonicity. Suppose $u \in X$. Assume $X \in N(u)$. Then there exists an open $\mathcal{O} \supseteq X$ s.t. if $\widehat{a} \subseteq \mathcal{O}$, then $\neg a \in u$. $u \in X$ implies $u \in \mathcal{O}$. Since the extension is compact and totally order-disconnected space, then $\mathcal{O} = \bigcup_{\widehat{a} \subseteq \mathcal{O}} \widehat{a}$. Hence $u \in \widehat{a}$ for some a. That means $a \in u$. Then $\bot \in u$ follows from $x \wedge \neg x \vdash \bot$. But this contradicts the fact that u is a prime ultrafilter. Therefore $X \notin N(u)$. □

N6 $x \vdash \neg\neg x$

This is so-called constructive double negation. For it we have the following proposition:

Proposition 27. *$x \vdash \neg\neg x$ corresponds to the condition: $\{t | \uparrow s \in N(t)\} \in N(s)$. And it is canonical.*

Proof. Obviously $x \vdash \neg\neg x$ corresponds the second-order condition: for any $s \in F$ and $X \subseteq \mathcal{U}(\mathbb{F})$ if $s \in X$, then $\{t \mid X \in N(t)\} \in N(s)$. Similar to the Sahlqvist formula we replace X with the minimal instantiation making the antecedent true, i.e. $\uparrow s$. It is easy to verify that the result is equivalent to the original second-order condition.
Now Consider canonicity. Take any u. For each v_i with $\uparrow u \in N(v_i)$, there is an open $\mathcal{O}_i \supseteq \uparrow u$ for any $\widehat{a_{ij}}$ with $\widehat{a_{ij}} \subseteq \mathcal{O}_i$ $\neg a_{ij} \in v_i$ by Definition 7. $\mathcal{O}_i \supseteq \uparrow u$ implies $u \in \widehat{a_{ik}}$ for some $\widehat{a_{ik}} \subseteq \mathcal{O}_i$ since $\mathcal{O}_i = \bigcup_{\widehat{a_{ij}} \subseteq \mathcal{O}_i} \widehat{a_{ij}}$. Now take $\bigcup \widehat{\neg a_{ik}}$. It

is open obviously. And $\{v_i \mid \uparrow u \in N(v_i)\} \subseteq \bigcup \widehat{\neg a_{ik}}$. Each clopen included in open $\bigcup \widehat{\neg a_{ik}}$ is some $\widehat{\neg a_{ik}}$. $u \in \widehat{a_{ik}}$ implies $a_{ik} \in u$. So $\neg\neg a_{ik} \in u$ by the axiom $x \vdash \neg\neg x$. Hence $\bigcup \widehat{\neg a_{ik}} \in N(u)$. Therefore $\{v \mid \uparrow u \in N(v)\} \in N(u)$. □

N7 $\neg\neg x \vdash x$

This is so-called classical double negation. For it we have the following proposition:

Proposition 28. $\neg\neg x \vdash x$ *corresponds to the condition:* $\{t \mid \overline{\downarrow s} \in N(t)\} \notin N(s)$.

Proof. Obviously $\neg\neg x \vdash x$ corresponds the second-order condition: for any $s \in F$ and $X \in \mathcal{U}(\mathbb{F})$ if $s \notin X$, then $\{t \mid X \in N(t)\} \notin N(s)$. Equivalently the condition is: if $s \in \bar{X}$ then $\{t \mid X \in N(t)\} \notin N(s)$. $\bar{X}$ is an ideal. Analogous to the previous proposition we substitute $\bar{X}$ with the minimal instance making the antecedent true, i.e. $\downarrow s$. This amounts to replace X with $\overline{\downarrow s}$. Then the result is: if $s \in \overline{\overline{\downarrow s}}$ then $\{t \mid \overline{\downarrow s} \in N(t)\} \notin N(s)$, i.e. $\{t \mid \overline{\downarrow s} \in N(t)\} \notin N(s)$ since $\overline{\overline{\downarrow s}} = \downarrow s$ and $s \in \downarrow s$. It is easy to check that the result is equivalent to the original condition.

The problem of canonicity of $\neg\neg x \vdash x$ is still open. □

Summery: We have discussed seven sequents. All of them except N7 are canonical. We summarize our results about them in a table as follows:

sequents	correspondence property
N1 $\top \vdash \neg\bot$	for any $s \in F, \emptyset \in N(s)$
N2 $\neg\top \vdash \bot$	for any $s \in F, F \notin N(s)$
N3 $\neg x \wedge \neg y \vdash \neg(x \vee y)$	N is reduced to the binary relation R_1 if $N(s) \neq \emptyset$ for each $s \in F$
N4 $\neg(x \wedge y) \vdash \neg x \vee \neg y$	N is reduced to binary relation R_2 if $N(s) \neq Full$ for each $s \in F$
N5 $x \wedge \neg x \vdash \bot$	if $s \in X$, then $X \notin N(s)$
N6 $x \vdash \neg\neg x$	$\{t \mid \uparrow s \in N(t)\} \in N(s)$
N7 $\neg\neg x \vdash x$	$\{t \mid \overline{\downarrow s} \in N(t)\} \notin N(s)$

6 the Relations between N and other Rs in literatures

In this section we will discuss the relation our semantics and others, which will show clearly that negation-neighborhood semantics is more powerful to talk about negation.

6.1 the Relations with $R_{\triangleright}$ and $R_{\triangleleft}$

In [14] the authors use $R_{\triangleright}$ and $R_{\triangleleft}$ to capture respectively the corresponding the connectives $\triangleright$ and $\triangleleft$. $\triangleright$ satisfies antitony, N1 and N3 in our context. $\triangleleft$ satisfies antitony, N2 and N4 in our context. The definitions of satisfaction concerning $\triangleright$ and $\triangleleft$ respectively are:

(g) $\mathbb{M}, w \Vdash \triangleright\, \alpha$ if and only if for all $v \in F$ with $wR_{\triangleright}v$ we have $\mathbb{M}, v \nVdash \alpha$.

(h) $\mathbb{M}, w \Vdash \triangleleft\, \alpha$ if and only if there is a $v \in F$ with $wR_{\triangleleft}v$ and $\mathbb{M}, v \nVdash \alpha$.

The component of the frame in [14] related to our discuss is $\langle F, \leqslant, R_{\triangleright}, R_{\triangleleft}\rangle$. $\langle F, \leqslant, \rangle$ is the same as ours. $R_{\triangleright}$ and $R_{\triangleleft}$ satisfies respectively:

(LEFT) $\geqslant \circ R_{\triangleright} \circ \leqslant \subseteq R_{\triangleright}$.

(RIGHT) $\leqslant \circ R_{\triangleleft} \circ \geqslant \subseteq R_{\triangleleft}$.

It should be pointed out that unlike our Up-frames, the frames in [14] use downward sets (See the remark below Definition 2). So $\geqslant$ and $\leqslant$ in above conditions should be reversed, i.e. in our setting $R_{\triangleright}$ should satisfies (RIGHT) and $R_{\triangleleft}$ should satisfies (LEFT). If we use Down-frames then $R_{\triangleright}$ matches still (LEFT) and $R_{\triangleright}$ does (RIGHT) as in [14]. The interested readers are asked to check it.

Now we look the corresponding of syntax and semantics in our background. That $\neg$ satisfies N3 enable us to reduce the definition of satisfaction concerning $\neg$ to (c_1). Furthermore that $\neg$ satisfies N1 corresponds $\emptyset \in N(s)$ in the frame, which means K is empty in the resulted R_1 frame by reduction. So we can reduces N to R_1 without adding K. In such case c_1 is:

$\mathbb{M}, s \Vdash \neg\alpha$ iff for any $\mathbb{M}, t \in F$ if $\mathbb{M}, t \Vdash \alpha$ then $t \in R_1(s)$.

Equivalently

$\mathbb{M}, s \Vdash \neg\alpha$ if and only if for any $t \in F$ if $t \notin R_1(s)$ then $\mathbb{M}, t \nVdash \alpha$.

Hence $R_{\triangleright}$ in [14] amounts to $\bar{R}_1$ here. Indeed we can verify $\bar{R}_1$ satisfies the above corresponding condition (RIGHT).

Similar analysis leads us to the result that $R_{\triangleleft}$ in [14] amounts to R_2 here. The verification is left to the readers. In [14] the authors also point out that if $\neg$ satisfies N1 to N4 then the relation can further degenerate to a function. In our setting we can verify the fact by showing that if t_1 and t_2 satisfy $s\bar{R}_1t_1$, sR_2t_1, $s\bar{R}_1t_2$ and sR_2t_2, then $t_1 = t_2$.

6.2 the Relations with perp, C and *

As we know, perp $(\bot)^2$ and star(*) are most eminent treatment of negation. In some literatures logicians do not use perp directly but the complement of perp, which is called compatible relation. A frame compatibility frame is a triple $\langle F, C, \leqslant\rangle$, where F and $\leqslant$ are the same as ours, and C is called compatible relation satisfying:

(C) For all $s, t \in F$ if $s' \leqslant, t' \leqslant t$ and sCt, then $s'Ct'$.

A star frame is different from a compatibility frame just in that the binary relation on F is replaced by a function * on F. * satisfies:

(*) If $s \leqslant t$, then $t^* \leqslant s^*$.

The semantic definition of negation by the two treatment are the following respectively:

- $(\neg^{\bot})$ $\mathbb{M}, s \Vdash \neg\alpha$ if and only if for any t if $t \Vdash \alpha$ then $t \bot s$

[2] People customarily use the same symbol to denote perp as the symbol of false. I follow the custom. I hope that the readers can distinguish them in the context.

- $(\neg^*)$ $\mathbb{M}, s \Vdash \neg\alpha$ if and only if $s^* \nVdash \alpha$

[18] uses compatibility frames to characterize $K_{\_}$, which amounts to S with N1 and N3 here. It is easy to see that R_1 is actually the perp and C is the $\bar{R}_1$ by verifying that $\bar{R}_1$ satisfies the above condition .

Using star frame [18] further characterizes K_s, i.e. S with N1, N2, N3 and N4. This is a case of relation degenerating to a function in our setting. As we mentioned in above subsection we can verify that the intersection of $\bar{R}_1$ and R_2 is a function. Here we can further verify that the function indeed satisfies (*). The readers are asked to check it.

One of the results in [16] is that N6 corresponds to that C is symmetric. It is not difficult to check that $\bar{R}_1$ is symmetric under the condition corresponding to N6, i.e. $\{t| \uparrow s \subseteq R_1(t)\} \subseteq R_1(s)$. Another result in [16] is that N2 corresponds to $\forall x \exists y (xCy)$. The condition is equivalent to the one in our setting: for any $s \in F$ $\bar{R}_1(s) \neq \emptyset$. Furthermore it means that $R_1(s)$ is not full , which is exactly what Proposition 15 means since N is downward closed. As last special case we see N5. [16] shows that N5 corresponds to the reflexivity on compatibility frame. In our setting $\{s\} \nsubseteq R_1(s)$ since $s \in \{s\}$ by Proposition 26. So $s \in \bar{R}_1(s)$, which means $\bar{R}_1$ is reflexive.

7 Duality between frame and algebra

In the last section we will establish the duality between the objects of DLNs and the objects of descriptive general NFs. We believe that the duality can be generalized to the one between the category of DLNS with homomorphisms and the category of descriptive general NFs with bounded morphisms although we have not yet the result for the present. First we give some relevant definitions.

Definition 29. *A general negation-neighborhood frame (GNF) is a tuple $\mathbb{G} = \langle \mathbb{F}, A \rangle$, where $\mathbb{F}$ is a NF and A is a subset which includes $\emptyset, F$ and closed under $\cup$, $\cap$ and the operation $\neg_N$ satisfying $s \in \neg_N X$ iff $X \in N(s)$.*

Let $\mathbb{G} = \langle \mathbb{F}, A \rangle$. We denote the topology having A as a basis by $\mathbb{X} = (F, \tau_A)$. The collections of open upsets and closed upsets of $\mathbb{X}$ is denoted by $\mathcal{O}(\mathbb{X})$ and $\mathcal{K}(\mathbb{X})$ respectively.

Definition 30. *Let $\mathbb{G}$ be a GNF. $\mathbb{G}$ is called differentiated if for all $s, t \in F$:*
$s \nleqslant t \Rightarrow \exists a \in A(s \in a$ *and* $t \notin a)$.
$\mathbb{G}$ is called tight if for all $s \in F$, all open upsets $\mathcal{O} \in \mathcal{O}(\mathbb{X})$ and all upsets u of F
$\mathcal{O} \in N(s) \Leftrightarrow \forall a \in A(a \subseteq \mathcal{O} \Rightarrow a \in N(s))$
$u \in N(s) \Leftrightarrow \exists \mathcal{O} \in \mathcal{O}(\mathbb{X})$ *s.t.* $u \subseteq \mathcal{O}$ *and* $\mathcal{O} \in N(s)$.
$\mathbb{G}$ is called compact if for all $a_i, b_j \subseteq A$ with $i \in I$ and $j \in J$
$\bigcap a_i \subseteq \bigcup b_j \Rightarrow \exists$ *finite* $I_0 \subseteq I$, *finite* $J_0 \subseteq J$ *s.t* $\bigcap a_i \subseteq \bigcup b_j$ *for* $i \in I_0$ *and* $j \in J_0$
We call $\mathbb{G}$ descriptive if $\mathbb{G}$ has all of above three properties.

Definition 31. *For a GNF $\mathbb{G} = \langle \mathbb{F}, A \rangle$, its dual is defined as $\mathbb{G}^* = \langle A, \cup, \cap, \emptyset, F, \neg_N \rangle$. For a DLN $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg \rangle$. its dual is defined as $\mathbb{A}_* = \langle \mathbb{A}_\bullet, \widehat{A} \rangle$, where $\widehat{A} = \{\widehat{a} \mid a \in F\}$, i.e. the collection of all clopen upsets of A's dual space.*

It is easy to see that $\mathbb{G}^*$ is a DLN and $\mathbb{A}_*$ is a descriptive GNF. Now we deal with the duality of morphisms. First we define the bounded morphism between GNFs and the homomorphism between DLNs. We fix $\mathbb{G} = \langle F, \leqslant, N, A \rangle$ to denote a GNF and $\mathbb{A} = \langle A, \vee, \wedge, \bot, \top, \neg \rangle$ to denote a DLN.

Definition 32. *A bounded morphism from $\mathbb{G}$ to $\mathbb{G}'$ is an order-preserving map $\theta : F \to F'$ satisfying the following two conditions:* (1) $\theta^{-1}[a'] \in A$ *for any* $a' \in A'$; (2) $\theta^{-1}[U'] \in N(s)$ *iff* $U' \in N'(\theta(s))$ *for any* $s \in F$ *and* $U' \in \mathcal{U}(F')$, *where* $\theta^{-1}[U'] = \{s \in F \mid \theta(s) \in U'\}$.

we call θ an embedding from $\mathbb{G}$ to $\mathbb{G}'$, written $\mathbb{G} \rightarrowtail \mathbb{G}'$, if θ is an injective bounded morphism from $\mathbb{G}$ to $\mathbb{G}'$ and satisfies the following condition: for any $a \in A$ there is an a' s.t. $\theta[a] = \theta[F] \cap a'$.

We call $\mathbb{G}'$ a bounded morphic image of $\mathbb{G}$, written $\mathbb{G} \twoheadrightarrow \mathbb{G}'$, if there is a surjective bounded morphism from $\mathbb{G}$ to $\mathbb{G}'$.

$\mathbb{G}$ and $\mathbb{G}'$ are called isomorphic, written $\mathbb{G} \cong \mathbb{G}'$, if there is a surjective embedding from $\mathbb{G}$ to $\mathbb{G}'$.

A homomorphism from $\mathbb{A}$ to $\mathbb{A}'$ is a map $\eta : A \to A'$ satisfying the following conditions: (1) $\eta(a \vee b) = \eta(a) \vee' \eta(b)$; $\eta(a \wedge b) = \eta(a) \wedge' \eta(b)$; (2) $\eta(\bot) = \bot'$; $\eta(\top) = \top'$; (3) $\eta(\neg a) = \neg' \eta(a)$.

We call η an embedding from $\mathbb{A}$ to $\mathbb{A}'$, written $\mathbb{A} \rightarrowtail \mathbb{A}'$, if η is an injective.

We call $\mathbb{A}'$ a bounded morphic image of $\mathbb{A}$, written $\mathbb{A} \twoheadrightarrow \mathbb{A}'$, if there is a surjective homomorphism from $\mathbb{A}$ to $\mathbb{A}'$.

Proposition 33. *$(\cdot)_*$ and $(\cdot)^*$ are dually equivalent between DLNs and descriptive GNFs.*

Proof. it suffices to prove that $\mathbb{A} \cong (\mathbb{A}_*)^*$ and $\mathbb{G} \cong (\mathbb{G}^*)_*$ if $\mathbb{G}$ is descriptive. Let $p : a \mapsto \widehat{a}$. The proof that p is an isomorphism from $\langle A, \vee, \wedge, \bot, \top\rangle$ to $\langle \widehat{A}, \cup, \cap, \emptyset, Pf\mathbb{A}\rangle$ is standard. We now just show that $p(\neg a) = \neg_{N_\neg}(p(a))$ for any $a \in A$ as follows:

$u \in p(\neg a)$ iff $u \in \widehat{\neg a}$ iff $\neg a \in u$ iff $\widehat{a} \in N_\neg(u)$ iff $u \in \neg_{N_\neg}(\widehat{a})$.

Now we turn to $\mathbb{G} \cong (\mathbb{G}^*)_*$ for descriptive $\mathbb{G}$. Let $\mathbb{G} = \langle F, \leqslant, N, A\rangle$, then $\mathbb{G}^* = \mathbb{A} = \langle A, \cup, \cap, \emptyset, F, \neg_N\rangle$ and $(\mathbb{G}^*)_* = \langle \mathcal{U}(Pf\mathbb{A}), \subseteq, N_{\neg_N}, \widehat{A}\rangle$. Let $q : s \mapsto U_s = \{a \in A \mid s \in a\}$. From differentiation and compactness of $\mathbb{G}$ it is easy to show that q is bijective. In order to show that q is a bounded morphism it suffices to show

(1) $q^{-1}[\widehat{a}] \in A$ for any $a \in A$;

(2) $q^{-1}[X] \in N(s)$ iff $X \in N_{\neg_N}(q(s))$ for any $X \in \mathcal{U}(Pf\mathbb{A})$;

(3) for any $a \in A$ there is $\widehat{b} \in A$ s.t. $q[a] = q[F] \cap \widehat{a}$.

In fact we have $U_s \in q[a]$ iff $s \in a$ iff $a \in U_s$ iff $U_s \in \widehat{a}$.

So $q[a] = \widehat{a}$. Then $q^{-1}[\widehat{a}] = a$ since q is bijective. So (1) follows from it immediately.

For (2) first we have

(Clopen) $\quad q^{-1}[\widehat{a}] \in N(s)$ iff $\widehat{a} \in N_{\neg_N}(q(s))$.

by the following inference:

$q^{-1}[\widehat{a}] = a \in N(s)$ iff $s \in \neg_N(a) \in U_s$ iff $\widehat{a} \in N_{\neg_N}(q(s))$.

We can verify that

$q[\mathcal{O}] = \bigcup_{a \subseteq \mathcal{O}} \widehat{a}$ for all $\mathcal{O} \in \mathcal{O}(\mathbb{X})$ and

$q^{-1}[\mathcal{P}] = \bigcup_{\widehat{a} \subseteq \mathcal{P}} a$ for all $\mathcal{P} \in \mathcal{O}(\mathbb{A}^\pi)$.

Then we have

(Intermediary) $\quad q[\mathcal{O}] \in \mathcal{O}(\mathbb{A}^\pi)$ iff $\mathcal{O} \in \mathcal{O}(\mathbb{X})$.

Therefore we can infer for any $X \in \mathcal{U}(Pf\mathbb{A})$ as the following

$q^{-1}[X] \in N(s)$ iff
$\exists \mathcal{O}(\mathbb{X})$ s.t. $q^{-1}[X] \subseteq \mathcal{O}$ and $\forall a \in A(a \subseteq \mathcal{O} \Rightarrow a \in N(s))$ iff
$\exists \mathcal{O}(\mathbb{X})$ s.t. $q^{-1}[X] \subseteq \mathcal{O}$ and $\forall a \in A(a \subseteq \mathcal{O} \Rightarrow \widehat{a} \in N_{\neg_N}(U_s))$ iff
$\exists \mathcal{O}(\mathbb{X})$ s.t. $q[\mathcal{O}] \subseteq X$ and $\forall a \in A(\widehat{a} \subseteq q[\mathcal{O}] \Rightarrow \widehat{a} \in N_{\neg_N}(U_s))$ iff
$\exists \mathcal{P} \in \mathcal{O}(\mathbb{A}^\pi)$ s.t. $X \subseteq \mathcal{P}$ and $\forall a \in A(\widehat{a} \subseteq \mathcal{P} \Rightarrow \widehat{a} \in N_{\neg_N}(U_s))$ iff
$X \in N_{\neg_N}(U_s)$

The first iff is by $\mathbb{G}$'s tightness; the second is by (Clopen); the third is by the fact that q is bijective; the fourth is by (Intermediary).

For (3) $q[a] = \widehat{a} = Pf\mathbb{A} \cap \widehat{a} = q[F] \cap \widehat{a}$.

□

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