Revision on Stable Sets*

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Abstract: This article focuses on the revision of stable sets which are considered elegant representations for full belief states of fully-introspective agents. Two stages of change of a given stable set and new information are distinguished and some work has been done in respective stages: one is to do the revision work from the stable set to an intermediate theory, and the other is to expand the intermediate theory to get some new stable sets and then select the best one with the help of information value. Three different perspectives within AGM traditions for the revision work from a stable set to an intermediate theory have been put forward. Maximal **S5** non-implying sets, sphere system and epistemic entrenchment orderings are employed in those three different approaches. The focus there is the contraction operator from stable sets (and new information) to intermediate theories. Some representation theorems between contraction operators and those three different systems are provided. Then the revision operator from stable sets to intermediate theories can be characterized with the help of contraction operators and a variant *Levi-identity*.

Key Words: Introspection, Stable sets, Contraction, Revision, Information value

1. Introduction

The system for the *classical epistemic logic* is **S5**, and *doxastic logic*, **KD45** respectively. The difference between them is that the former contains axiom schema **T** (**K**A→A) while the latter does not contain **B**A→A in general since the belief may be false. Semantically we know that the binary relation of a Kripke frame which represents **S5** is an *equivalent* relation, and for **KD45**, is *serial*, *transitive* and *Euclidean*. Commonly we think that **KD45** is an appropriate logical system to represent beliefs of fully introspective agents. But it is clear that the system cannot deal with the phenomenon of belief change which is common in realities.

The classical static epistemic (doxastic) logic must be changed or updated in order to characterize the phenomenon of epistemic (belief) change and the processes of those changes in particular. In recent three decades, different approaches have been put forward to fulfill the trends which is called dynamic turn (cf. [vBen96], [vBen03], [vBen05]). The main two streams in this turn are classical belief revision theories (which are developed by Alchourrón, Gärdenfors, Markinson) (cf. [AM82], [AGM85], [Gär88]) and dynamic logics (including dynamic epistemic logics and dynamic doxastic logics, update logics, and etc.) (cf. [vBen96]; [Har84], [KHT00]; [Ger99], [GG97]; [Seg95], [Seg97], [Seg98], [Seg99]). We will not explore all the dynamic contexts in this paper, but only focus on a special problem of belief revision.

In Lindström and Rabinowicz [LR97, 99, 0x], according to introspection and dynamism, doxastic agents can be divided into five different types: non-introspective static agents, introspective static agents, non-introspective dynamic agents, introspective dynamic agents whose doxastic inputs are limited to propositions about the external world, introspective dynamic agents whose doxastic inputs may contain propositions about the agent's own belief states. For the last one, any dynamic doxastic sentence can be taken as a doxastic input and such sentences may also be contained in an agent's belief sets. The last two types are becoming the main interests of researchers in different areas.

We expect to find a special kind of sets which can reflect the fully introspective property of ideal rational agents. There is a kind of sets which are closed under positive introspection, negative introspection and logical consequence. They are called stable sets [Sta80] in computer science, and the study of such theories is considered part of *autoepistemic logic* (AEL) introduced

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by Moore [Moo84, 85, 88]. Some philosophers also call a subset of such theories saturated sets [ACo99]. Since such theories can be taken as sound belief states for rational fully introspective agents, it is reasonable to do revision work on stable sets in order to reflect the revision of introspective agents.

The main work of this article is to do the revision on general stable sets. Given a stable set and new information, there are two processes in the whole belief change: one is to do the revision work from the stable set to an intermediate theory, and the other is to expand the intermediate theory to get some new stable sets and then select the best one based on some principles. In the next two sections of this article, we will briefly introduce the relevant notions of stable sets and their properties provided by Moore, Halpern & Moses [HM84], Konolige [Kon88, 93], Marek & Truszynscki [MT93] and [ACo99] et.al; and then try to analyze the difficulties in revising stable sets, some possible approaches (but have restrictions) will be brought out to show what the difficulties lie in. In section 4-5, we will provide three different perspectives within AGM traditions for the revision work from a stable set to an intermediate theory. Maximal **S5** non-implying sets, sphere system and epistemic entrenchment orderings are employed in those three different methods. The focus of them is the contraction operator from stable sets (and new information) to intermediate theories. Some representation theorems between contraction operators and those three different systems are provided. Then the revision operator from stable sets to intermediate theories can be characterized with the help of contraction operators and a variant *Levi-identity*. Section 6 is going to present a selection method within the stable expansions of a given intermediate theory. The last section is conclusion and further work.

2. Stable sets and their properties

Definition 1 (Stalnaker): Call a belief set Γ stable if it satisfies:

(1) Γ is closed under tautological consequence.

(2) If $\varphi\in\Gamma$, then $\mathbf{B}\varphi\in\Gamma$.

(3) If $\varphi\notin\Gamma$, then $\neg\mathbf{B}\varphi\in\Gamma$.

The language L there augments the classical propositional language L_0 with modal operator **B** (The original modal operator is L, for we are going to explore belief revision, belief operator **B** is employed instead). By [Sta80], it shows that there can be a stable set that is inconsistent since the set of all formulae (in language L) satisfies the above three conditions. It can be derived from the definition that for every stable set Γ, $\mathbf{B}\varphi\in\Gamma$ or $\neg\mathbf{B}\varphi\in\Gamma$, that is, any stable set is complete with respect to the belief literals.

Definition 2 (Arló Costa): Call a set σ saturated if it satisfies the following two constraints:

(A1) $A\in\sigma$ iff $\mathbf{B}A\in\sigma$.

(A2) $A\notin\sigma$ iff $\neg\mathbf{B}A\in\sigma$.

Saturated sets describe some kind of states for full belief. In [ACo99], "commitment to full belief is mirrored by commitment to accept as true; and commitment not to accept is mirrored by commitment not to fully believe". According to the definition 1-2, it is clear that every saturated set σ is a stable set. And actually by [MT93] (it is easy to prove), if only consistent stable sets are mattered, then conditions (1)-(3) in definition 1 are equivalent to A1 and A2 in definition 2.

Definition 3 (AE Valuation: $\vDash_{I,\Gamma}$): For any formula $A\in L$ we have the following conditions:

$\vDash_{I,\Gamma}\varphi$ iff $\varphi\in I$ (if φ is an objective formula)

$\vDash_{I,\Gamma}\mathbf{B}\varphi$ iff $\varphi\in\Gamma$

It was originally introduced by [Kon88]. I is a set of objective formulae (without belief operators in any formulae) and Γ is a subset of L. It is clear that Γ makes this valuation different from the classical one. We call such kind of valuation $\vDash_\Gamma$ valuations with modal index.

Definition 4 (Konolige): Any set of sentences T which satisfies the equation

$T=\{\varphi|X\vDash_T\varphi\}$ is an autoepistemic extension (briefly AE extension) of X (X is a base set). Another semantic characterization of AE extensions is as the following: A set T is an AE extension of X iff it satisfies the equation $T=\{\varphi|X\cup\mathbf{B}T\cup\neg\mathbf{B}\overline{T}\vDash\varphi\}$, where $\mathbf{B}T$ stands the set of formulae $\{\mathbf{B}\varphi|\varphi\in T\}$,

$\neg\mathbf{B}\overline{T}$ stands the set of formulae $\{\neg\mathbf{B}\varphi|\varphi\notin T\}$. We have $\mathbf{B}T\cup\neg\mathbf{B}\overline{T}\vDash\varphi$ iff $\vDash_T\varphi$. Call an extension *weakly grounded* if it obeys the equation $T=\{\varphi|X\cup\mathbf{B}T\cup\neg\mathbf{B}\overline{T}\vDash\varphi\}$.

Definition 5 (Moore): A set of sentences T of L is sound with respect to the premises X if every AE valuation of T that is a model of X is also a model of T. T is semantically complete if T contains every sentence that is true in every AE model of T. If T is sound and complete with respect to X, it is called a stable expansion of X.

Stable expansions are exactly AE extensions.

Proposition 1 (Marek & Truszynscki): Suppose an agent has only objective sentences in her base set X. These sentences determine *a unique extension* for the agent. That is, if X is a set of objective sentences, it has exactly one AE extension T of X.

We may consider the relation between stable sets and AE extensions. It is clear that every AE extension of X is a stable set containing X. The converse does not hold in general, but a partial converse is available if we consider stable sets as AE extensions of their own objective sentences.

Proposition 2 (Moore): Every stable set S is an AE extension of S_0 ($S_0=L_0\cap S$) (L_0 represents the set of all objective sentences).

Proposition 3 (Moore): If two stable sets agree on objective sentences, they are equal.

Proposition 4 (Konolige): Let W be a set of objective sentences closed under tautological consequence. There is a unique stable set S such that S_0=W. W is called the kernel of the stable set.

Proposition 5 (Marek & Truszynscki): If two stable sets S and T satisfy $S\subseteq T$ or $T\subseteq S$, then S=T.

Definition 6 (Konolige, Marek & Truszynscki): An AE extension T of X is minimal for X if there is no stable set S containing X such that $S\subset T$.

Definition7 (Konolige, Marek & Truszynscki): A stable set S is minimal for X if S contains X and there is no other stable set S' containing X such that $S'_0\subset S_0$.

It is clear that every minimal AE extension for X is a minimal stable set for X. And if a minimal stable set for X is an AE extension of X, it is a minimal extension.

The following three propositions are all from Konolige, they discuss *normal forms* for belief sentences.

Proposition 6 : Every sentence of L_1 (L_1 means all the sentences with their belief degree not more than 1) is equivalent to a sentence of the form $(L_1\vee\omega_1)\wedge(L_2\vee\omega_2)\wedge\ldots\wedge(L_n\vee\omega_n)$, where each L_i is a disjunction of belief literals on objective sentences, and each ω_i is objective.

Proposition 7 : Every belief atom $\mathbf{B}\varphi$, where φ is from L_1, is equivalent to a sentence of L_1.

Proposition 8 : Every set of T of L-sentences has a **K45**-equivalent set in which each sentence is of the form $\neg\mathbf{B}\alpha\vee\mathbf{L}\beta_1\vee\ldots\vee\mathbf{B}\beta_n\vee\omega$, with α, β_i, and ω all being objective sentences. Any of the disjuncts, except for ω, may be absent.

Proposition 9 (Halpern & Moses): Let S be a stable set. Then it holds for all sentences A, $C\in L_1$ that

(1) $\mathbf{B}A\vee C\in S$ iff $A\in S$ or $C\in S$.

(2) $\neg\mathbf{B}A\vee C\in S$ iff $A\notin S$ or $C\in S$.

Proposition 10 (Arló Costa): Let S be a stable set. Then for any $A\in$Thm(**S5**) (meaning the set of **S5** theorems), $A\in S$. That is, every stable set contains all **S5** theorems.

3. Problems of revision on stable sets

3.1. Brief introduction to belief revision

It is well known that there are three different types of belief change about static worlds for rational agents in AGM tradition: expansion, contraction and revision.

Expansion: the agent adds new belief A to her old belief set without giving up any old beliefs. If G is her old belief set, then G+A denotes the belief set that results from expanding G with A. Formally, $G+A=Cn(G\cup\{A\})$. It is clear that G+A need to be consistent if rational agents are considered.

Contraction: The agent gives up a proposition A which was formerly believed. This often requires the agent to give up more her old beliefs (that logically implies A). We use $G \div A$ to denote belief set of the agent after contracting G with A.

Revision: The agent accepts the new information A first (priority of new information) and the new belief set should be kept consistent.

Additionally, the change of old beliefs in order to incorporates A should be minimal. We use G^*A to denote belief set after revising G with A.

It can be seen that expansion is simple and the other two are relatively complex. Actually contraction and revision can be seen as two sides of a coin with the following bridges:

The Levi Identity: $G^*A= (G \div \neg A) +A$

The Harper Identity: $G \div A= (G^*A) \cap (G^* \neg A)$

Syntactically postulates of revision are given as the following:

(R1) $Cn(G^*A)=G^*A$ (*Closure*)

(R2) $A \in G^*A$ (*Success*)

(R3) $G^*A \subseteq G+A$ (*Inclusion*)

(R4) if $\neg A \notin G$ then $G \subseteq (G^*A)$ (*Preservation*)

(R5) if $\bot \notin Cn(\{A\})$ then $\bot \notin G^*A$ (*Consistency*)

(R6) if $\vdash A \leftrightarrow B$ then $G^*A=G^*B$ (*Extensionality*)

(R7) $G^*(A \wedge B) \subseteq G^*A+B$ (*Conjunctive inclusion*)

(R8) if $\neg B \notin G^*A$, then $(G^*A)+B \subseteq G^*(A \wedge B)$ (*Conjunctive vacuity*)

The first six postulates are basic and the last two are additional.

In order to understand those postulates more intuitively, some semantic counterparts are established such as *sphere system* and *epistemic entrenchment orderings*. The former was put forward by Adam Grove [Gro88]. The latter was introduced by Gärdenfors and Makinson [GM88].

3.2. Troubles appear when applying traditional postulates

Next we analyze the trouble in contracting and revising stable sets applying traditional AGM postulates. First let us consider the problems in contraction: Since rational introspective agents are concerned, contraction means giving up some sentences from the stable set the agent originally has. Since she can have a stable set from some premise before, it is reasonable for her to have another stable set after giving up some sentences. We assume all agents have an ability to derive a stable set automatically when accepted a set of sentences.

If we want to satisfy *stability* (that means every belief state of an agent is stable), the postulate *inclusion* may not hold in general. Because some forms of disbeliefs in the contracted information will appear in the new theory after contraction, but it is quite possible that they are not in the original theory. They are just knowledge of the contracted information in the original theory before contraction. Let S be a stable set and A a sentence. $\neg \mathbf{B}A \in S \div A$ but $\neg \mathbf{B}A \in S$ may not hold. It means $S \div A \subseteq S$ does not hold. Then we consider revision. The *stability* property is also assumed. Consider a stable set T* with its ground theory $T=Cn(\{\mathbf{B}\neg broken \rightarrow runs\})$. If the agent receives the new information $\neg broken$, then she believes her car is not broken and by logical implication she also gets the car runs. So it is obviously she will believe that her car runs by positive introspection and logical consequence, that is, $\mathbf{B}runs$ is in the new stable set T*'. But we know that the atom *runs* is not in the stable set T*, then $\neg \mathbf{B}runs$ should be in T* by negative introspection. And obviously *broken* is not in T*, so by R4 (*preservation*), T* is included in the new stable set T*' after revising by $\neg broken$. This means $\neg \mathbf{B}runs$ is also in T*'. It shows the belief state of the agent becomes inconsistent just because of receiving $\neg broken$. This is unreasonable, so postulate R4 must be discarded.

We also found that the *Levi Identity* and *Harper Identity* become invalid when stable sets are considered. So the bridge between contraction and revision as in AGM traditions collapses. Furthermore, we have only considered the objective new information used for contracting or revising belief states, it is quite possible for an agent to receive some new information about her own belief state and then make a revision.

We may also first do revision to obtain an intermediate theory and then expand it to a desired stable set. Classical postulates may be applied in the revision period. But the crucial problem there is *maintaining consistency* and *keeping the information loss to minimal* cannot be always implemented. For examples, A∧¬**B**A may be in the revised intermediate theory. And it is inappropriate for us to do revision work on the objective parts of stable sets to obtain objective intermediate theories and then to derive respective stable sets either, although every objective part decides a unique stable set.

3.3. Revising on objective parts of stable sets

Since we know that every stable set is determined by its objective part, it is intuitive to have an idea to do the revision work on the objective part of a stable set with an objective formula. If for every objective theory X, we can obtain a unique stable set with its objective part exactly X, it seems reasonable for revising those stable sets which are derived from objective bases.

Proposition 11: For an arbitrary objective theory $K\subseteq L_0$, there is a stable theory S ($S\subseteq L$) that is generated from K such that $K\subseteq S$ and $S\cap L_0=K$.

However, as we know, there are some stable sets which are derived from belief bases containing belief formulae. Then there will be some information loss if we do revision work in the above way. For example, a stable set S contains **B**A→C but A∉S and C∉S and A→C∉S. And new information A is employed to revise S. It is expected to obtain a new stable theory S' containing C. But C may not be in the objective part after revising the objective part of S with A since A→C are not contained in S. So the new stable theory derived from the revised objective part may not contain C.

So we hope to find some more general revision approaches. In the following several sections, three different perspectives in revising stable sets, such as 'sphere system', will be presented.

4. Revising stable sets applying maximal S5 non-implying sets

4.1. Postulates for revision on stable sets

Before presenting the main idea, we are going to define two preliminary notions slightly different from classical propositional logic. That is, the notion of consequence relation $\vdash_{\mathbf{S5}}$ (and $\vdash_N$) and the notion of *positive introspective consistent* (briefly, *PI-consistent*). First we are going to have a journey of revision on a stable set like Hansson's [Han99] style.

Definition 8 (Consequence relation $\vdash_{\mathbf{S5}}$): It is exactly like consequence relation ⊢ in classical propositional logic except all **S5** theorems are added as premises, that is, $\vdash_{\mathbf{S5}}A$ for every $A\in Thm(\mathbf{S5})$. And like operator Cn in classical logic, $Cn_{\mathbf{S5}}$ is used to reflect the consequence relation $\vdash_{\mathbf{S5}}$. It is assumed to satisfy the following properties.

$\Sigma\subseteq Cn_{\mathbf{S5}}(\Sigma)$	(Inclusion)
If $\Sigma\subseteq\Gamma$, then $Cn_{\mathbf{S5}}(\Sigma)\subseteq Cn_{\mathbf{S5}}(\Gamma)$	(Monotony)
$Cn_{\mathbf{S5}}(Cn_{\mathbf{S5}}(\Sigma))=Cn_{\mathbf{S5}}(\Sigma)$	(Iteration)
If $A\in Cn_{\mathbf{S5}}(\Sigma\cup\{C\})$, then $A\to C\in Cn_{\mathbf{S5}}(\Sigma)$	(Deduction)
If A can be derived from $\Sigma\cup Thm(\mathbf{S5})$ in classical logic, then $A\in Cn_{\mathbf{S5}}(\Sigma)$	(Supraclassicality)
If $A\in Cn_{\mathbf{S5}}(\Sigma)$, then $A\in Cn_{\mathbf{S5}}(\Sigma')$ for some finite Subset $\Sigma'\subseteq\Sigma$	(Compactness)

For every formula A, we use $Cn_{\mathbf{S5}}(A)$ as an abbreviation of $Cn_{\mathbf{S5}}(\{A\})$.

It is clear that *Necessitation rule* (RN) is not allowed in general in reasoning with $\vdash_{\mathbf{S5}}$, for example, $\mathbf{B}p\vdash_{\mathbf{S5}}\mathbf{BB}p$, but $p\nvdash_{\mathbf{S5}}\mathbf{B}p$ (p is an atomic sentence). Actually $Cn_{\mathbf{S5}}(\Sigma)=Cn(\Sigma\cup Thm(\mathbf{S5}))$ for any $\Sigma\subseteq L$.

Definition 9: Another consequence relation $\vdash_N$ is also introduced here. It is like $Cn_{\mathbf{S5}}$ except (RN) is added as an additional derivation rule. We use operator Cm to reflect $\vdash_N$. It is clear that Cm does not satisfy *deduction* since we know $A\vdash_N\mathbf{B}A$ but $\nvdash_N A\to\mathbf{B}A$ does not hold in general.

Definition 10 (*PI-consistent*): Say a set of formulae Σ in L is *PI-consistent* if $\perp\notin$Cm(Σ), otherwise it is *PI-inconsistent.* For an example, {p, **B**p} is *PI-consistent*, but {A, ¬**B**A} is *PI-inconsistent.* It is easy to verify that if a set of formulae is *PI-consistent*, then it is consistent.

Like AGM revision postulates, we hope the revision operator # on stable sets satisfies following basic properties. Let S be a stable set and A an arbitrary formula from language L which is expanded from the classical propositional language L_0.

(S1) If A is *PI-consistent*, then S#A is consistent. (*Consistency*)

(S2) A∈S#A. (*Success*)

(S3) S#A is a stable set. (*Stability*)

(S4) If ¬**B**A∉S, then S#A=S. (*Vacuity*)

(S5) If Cm(A)=Cm(C), then S#A=S#C. (*Extensionality*)

It is clear the postulate *closure* as in AGM revision can be derived from *stability*. In general, we do not expect the postulate *inclusion* S#A⊆Cn(S∪{A}) to hold in the occurrence of the above five postulates. Suppose it holds and S#A≠Cn(S∪{A}). We may find a formula C such that C∈S but C∉S#A. By *stability*, S and S#A are stable, so ¬**B**C is in S#A and **B**C is in S. Then **B**C and ¬**B**C will be both in Cn(S∪{A}). This means Cn(S∪{A}) is inconsistent or identical to S#A in order to satisfy *inclusion*. Since we know every consistent theory is contained in an inconsistent theory or contained in itself, the postulate *inclusion* becomes trivial.

Those postulates seem intuitive. And actually we are interested in consistent stable sets. For those inconsistent theories, we may just expand the new information (if it is *PI-consistent*) to get a consistent stable set. It is trivial. But how can we get the corresponding stable set *S*#A, which satisfies the above basic postulates, after the revision on an arbitrary consistent stable set S by the new information A. The interesting case concerning revision is that ¬**B**A is in S. We know every stable set is complete with respect to belief literals. Then if ¬**B**A is not in S, **B**A would be in S. And since every **S5** theorem is in S, A should be in S too. So we expect S#A=S. It is trivial. But if ¬**B**A is in S, we need to discard some formula first in order to maintain *PI-consistency*. Even we get a new theory that is *PI-consistent* containing new information A, it cannot be guaranteed that the new theory is stable.

4.2. Revision on stable sets to obtain an intermediate theory

4.2.1 Maximal S5 non-implying sets

An intuitive idea to obtain a new stable set S#A is as follows: do revision work (similar as classical AGM revision) on S to get a *PI-consistent* theory S' which contains A, then expand S' to obtain a consistent stable set containing S'. For the revision part, we may consider *partial meet contraction* similarly as in AGM tradition as its contraction side. Here is a notion of maximal non-implying subsets of $S\perp_{S5}A$ based on operator Cn_{S5}.

Definition 11 (maximal S5 non-implying subsets): A set of maximal **S5** non-implying sets $S\perp_{S5}A$ consists of all subsets S' of S such that each of them satisfies the following conditions [AM82]:

(1) S'⊆S

(2) A∉Cn_{S5}(S')

(3) For any S'' such that S'⊂S''⊆S, A∈Cn_{S5} (S'').

Intuitively that each subset S' is maximal with respect to inclusion which fails to imply A with the help of Thm(**S5**). It can be extended to the case $S\perp_{S5}T$ where T is a finite set of sentences and the second condition should be changed to Cn_{S5}(S')∩T=∅; the clause A∈Cn_{S5}(S'') in the third condition should be changed to Cn_{S5}(S')∩T≠∅. It is not difficult to verify that every such S' contains all **S5** theorems.

Definition 12 (relative Cn_{S5}-closure): For any two sets S and T of sentences, S is T-Cn_{S5}-closed if and only if Cn_{S5}(S)∩T⊆S.

Proposition 12: If S_1 is an S_2-Cn_{S5}-closed subset of S_2, and S_2 an S_3-Cn_{S5}-closed subset of S_3, then S_1 is an S_3-Cn_{S5}-closed subset of S_3.

Corollary 1: If S_1 is an S_2-Cn_{S5}-closed subset of S_2, and S_2 is closed under Cn_{S5}, then S_1 is closed under Cn_{S5}.

Proposition 13: If X∈$S\perp_{S5}T$, then X is S-Cn_{S5}-closed.

Upper bound property (similarly as in [AM82]): If $X\subseteq S$, and $Cn_{S5}(X)\cap T=\emptyset$, then there is some X' such that $X\subseteq X'\in S\perp_{S5}T$.

Proposition 14: The following two conditions are equivalent:

1. $S\perp_{S5}A= S\perp_{S5}C$
2. For all subsets T of S: $A\cap Cn_{S5}(T)=\emptyset$ if and only if $C\cap Cn_{S5}(T)=\emptyset$.

Proposition 15: If $X\in S\perp_{S5}T$ and $T\neq\emptyset$, then

$$X=\bigcap\{Y|\ X\subseteq Y\in S\perp_{S5}C \text{ for some } C\in T\}.$$

Proposition 16: S is a stable set and $X\in S\perp_{S5}T$, then X is closed under Cn_{S5}.

Proposition 17: Let S be a stable set and T a finite-base set. If $X\in S\perp_{S5}T$ and $C\in S$ then $X\cup T\vdash_{S5}C$.

Proposition 18: Let *S* be a stable set. If $X\in S\perp_{S5}A$, then $X\in S\perp_{S5}C$ for all $C\in S\backslash X$.

Proofs for those above results are similar as in [Han99], we omit all the details.

Using the properties of maximal **S5** non-implying subsets, the different kinds of contractions can be defined in many ways. If we want a *maxichoice contraction* as in AGM, then just take an arbitrary element from $S\perp_{S5}A$. And *full meet contraction* is defined $S\div A=\bigcap(S\perp_{S5}A)$. As showed in AGM tradition, maxichoice contraction is too conservative; too less information has been discarded. And full meet contraction is in the opposite; too much information has been lost. The result of compromising *informational economy* with *indifference* (Rott & Pagnucco 1999) [RP99] is to be born *partial meet contraction*. Like in [AGM85], we define a selection function and partial meet contraction with the help of $S\perp_{S5}A$.

Definition 13 (selection function): Let S be a set of sentences. A selection function for S is a function γ such that for all sentences A:

1. If $S\perp_{S5}A$ is non-empty, then $\gamma(S\perp_{S5}A)$ is a non-empty subset of $S\perp_{S5}A$, and
2. If $S\perp_{S5}A$ is empty, then $\gamma(S\perp_{S5}A)=\{S\}$.

Definition 14 (partial meet contraction): Let *S* be a set of sentences and γ a selection function for *S*. The partial meet contraction on S that is generated by γ is the operation $\sim_\gamma$ such that for all sentences A:

$$S\sim_\gamma A=\bigcap\gamma(S\perp_{S5}A).$$

An operation $\div$ on *S* is a partial meet contraction if and only if there is a selection function γ for *S* such that for all sentences A: $S\div A=S\sim_\gamma A$.

4.2.2. Postulates for contraction on stable sets to obtain an intermediate theory

Now we may use contraction procedure as the first step for revising a stable set S although the result theory may not be stable. We hope the contraction operator $\div$ satisfies the following postulates:

(C1) $S\div A=Cn_{S5}(S\div A)$. (*Closure*)

(C2) If $A\notin Cn_{S5}(\emptyset)$, then $A\notin S\div A$. (*Success*)

(C3) $S\div A\subseteq S$. (*Inclusion*)

(C4) If $A\notin S$, then $S\div A=S$. (*Vacuity*)

(C5) $S\subseteq Cn_{S5}((S\div A)\cup\{A\})$. (*Recovery*)

(C6) If $Cn_{S5}(A)=Cn_{S5}(C)$, then $S\div A=S\div C$. (*Extensionality*)

There are also two additional postulates similarly as in AGM literature:

(C7) $S\div A\cap S\div C\subseteq S\div(A\wedge C)$. (*Conjunctive overlap*)

(C8) If $A\notin S\div(A\wedge C)$, then $S\div(A\wedge C)\subseteq S\div A$. (*Conjunctive inclusion*)

Next we are going to show the representation theorem between partial meet contraction and the above fundamental first six postulates. In order to prove the theorem, some relevant basic postulates and their results need to be considered.

Relevance: If $C\in S$ and $C\notin S\div A$, then there is a set S' such that $S\div A\subseteq S'\subseteq S$ and that $A\notin Cn_{S5}(S')$ but $A\in Cn_{S5}(S'\cup\{C\})$.

Relative closure: $S\cap Cn_{S5}(S\div A)\subseteq S\div A$.

Uniformity: If it holds for all subsets S' of S that $A\in Cn_{S5}(S')$ if and only if $C\in Cn_{S5}(S')$, then $S\div A=S\div C$.

Proposition 19: If an operator $\div$ for a stable set S satisfies *relevance*, then it satisfies *relative closure*.

Proposition 20: If ÷ satisfies *inclusion* and *relative closure*, then it satisfies *closure*.

Proposition 21: If ÷ satisfies *uniformity*, then it satisfies *extensionality*.

Proposition 22: If an operator ÷ for a stable set S satisfies *extensionality* and *vacuity*, then it satisfies *uniformity*.

With the help of above results, we may speculate that the operator ÷ defined from *partial meet contraction*, if and only if it satisfies all the six fundamental postulates. The answer seems positive.

Theorem 1: The operator ÷ is an operator of partial meet contraction for a stable set S if and only if it satisfies the postulates of *closure, success, inclusion, vacuity, recovery* and *extensionality*.

For proofs of proposition 19-22 and theorem 1, please refer to [Han99].

Like in AGM tradition, we have shown that *partial meet contraction* operator characterize the rational postulates in contracting a stable set to get a new belief theory (it may not be stable). There is a 1-1 correspondence between *partial meet contraction* operator and basic 6 rational postulates. We know in classical belief revision theory, *recovery* is a controversial postulate, some operators such as *severe withdrawal* function do not satisfy *recovery*. But interestingly, their corresponding revision operators defined via *Levi identity* are equivalent in classical belief revision theory. We will explore this subject in *sphere system* and *epistemic entrenchment relation* in next section. Now we are first going to show some properties of revision operator for the stable sets via a variant type of *Levi identity*. In classical belief revision theory, for simple expansion of a theory, only logical closure is considered. But we need to consider *positive introspection closure* here.

Definition 15 (Positive introspection-expansion): Let *S* be a belief set and A a sentence. S+A (*S* expanded by A), the *positive introspection-expansion* (briefly *PI-expansion*) of S by A, is defined as follows:

S+A=Cm(S∪{A}).

Note that if ¬**B**A is in S, then S+A will be inconsistent.

Now we are ready to consider the revision on a stable set to obtain a *PI-consistent* intermediate theory with the help of *contraction* and *PI-expansion*. First we need to obtain an appropriate theory (after contraction) for *PI-expansion*. Suppose we have a stable set S and new information A. An intuitive idea is to give up ¬**B**A from S and then do *PI-expansion* with A. But we know S÷¬**B**A may be not closed under *PI-closure*. It is possible that ¬**B**A∈Cm(S÷¬**B**A). For example, if C and **B**C→¬**B**A are in S÷¬**B**A, then ¬**B**A∈Cm(S÷¬**B**A). So S÷¬**B**A may not be used as an appropriate theory for *PI-expansion* since S÷¬**B**A+A may be inconsistent. We need to exclude all such examples. A straightforward way is to take intersection with **B**S, that is, we take (S÷¬**B**A)∩**B**S as the object for *PI-expansion*. Where **B**S={**B**A: A∈S}. It is clear that **B**S⊆S since S is closed under Cm. Then Cn_{S5}(**B**S)⊆Cn_{S5}(S)=S since Cn_{S5} is monotonic. And it is not difficult to verify that S=Cn_{S5}(S)=Cn_{S5}(**B**S). As we know for every A∈S, then **B**A∈**B**S, and **B**A→A is an **S5** theorem. It follows that A∈Cn_{S5}(**B**S), that is, Cn_{S5}(S)⊆Cn_{S5}(**B**S). Hence we have S=Cn_{S5}(**B**S). Since the intermediate theory we want to obtain is an **S5** theory and the new theory after revision is stable, **B**S has the equivalent information value as S for introspective agents. It seems there is no informational loss for restricting S÷¬**B**A in **B**S at this level. And it can be guaranteed that ¬**B**A∉Cm((S÷¬**B**A)∩**B**S) if ¬**B**A is not an **S5** theorem.

Next we may apply this variant *Levi identity* similarly as in AGM tradition to build a bridge between revision and contraction with the help of *PI-expansion*.

Definition 16 (Variant Levi identity): Let *, ÷, + stand for revision, contraction and *PI-expansion* respectively. The revision procedure on a stable set S by A can be summarized in the form of an equation:

S*A=((S÷¬**B**A)∩**B**S)+A.

The operator * on S is an operator of *partial meet revision* if and only if there is some operator of partial meet contraction on S such that for all sentences A:

S*A= ((S$\sim_\gamma$¬**B**A)∩**B**S)+A.

Next we are going to consider the properties of the revision operator *. It is more general than AGM classical revision operator. So we hope to have some new properties that the original one may not have.

4.2.3. Fundamental Postulates for revision on stable sets to obtain intermediate theories

Like in AGM tradition, we may hope to have the following postulates for revising stable sets to obtain intermediate theories which will be used to expand corresponding new stable sets.

(M1) S*A=Cm(S*A). (*PI-Closure*)

(M2) If A is *PI-consistent*, then S*A is consistent. (*Consistency*)

(M3) A∈S*A. (*Success*)

(M4) S*A⊆S+A. (*Inclusion*)

(M5) If ¬**B**A∉S, then S*A=S. (*Vacuity*)

(M6) If Cm(A)=Cm(C), then S*A=S*C. (*PI-Extensionality*)

Observe that if * satisfies M1-M6, then it satisfies classical AGM postulates of *closure*, *success*, *vacuity* and *extensionality*. But the classical *consistency*, *inclusion* may fail. We know classical *consistency* of A cannot guarantee S*A to be consistent. And S*A is in Cm(S∪{A}) does not mean S*A be in Cn(S∪{A}) (this shows classical *inclusion* failed). Note *PI-Extensionality* here is slightly different from classical ones. We hope to obtain the same theory after revising same theory with *PI-equivalent* new information. Say two sets X and Y are *PI-equivalent* if and only if Cm(X)=Cm(Y).

Proposition 23: If a finite set $\sum$ consists of all belief atoms and A is a formula, then $\sum\vdash_N$ A if and only if $\sum\vdash_{\mathbf{S5}}$ A.

Theorem 2: The revision operator * satisfies M1-M6 if it is a partial meet revision.

5. Sphere system and epistemic entrenchment relation

5.1 Postulates for *severe withdrawal function* on stable sets to obtain intermediate theories

We know in AGM tradition, there are also some additional postulates for characterizing contractions with complex information, such as C7 and C8 in section 4. Similarly we hope to have all postulates for *severe withdrawal* function (on stable sets) as in Rott [Rot91] and Rott & Pagnucco [RP99]. The most obvious difference with AGM contraction function is presented by the absence of *recovery*. Let ∸ denote *severe withdrawal* operator. We hope it to have the following postulates. And of course the theories we got after contracted by *severe withdrawal* function are used by *Levi identity* (variant version) for obtaining the intermediate theories closed under Cm, which is revision equivalent to AGM contractions and it will be used for extending respective stable sets.

(C'1) S∸A=$Cn_{\mathbf{S5}}$(S∸A). (*Closure*)

(C'2) If A∉$Cn_{\mathbf{S5}}$(∅), then A∉S∸A. (*Success*)

(C'3) S∸A⊆S. (*Inclusion*)

(C'4) If A∉S or A∈$Cn_{\mathbf{S5}}$(∅), then S⊆S∸A. (*Vacuity*)

(C'6) If $Cn_{\mathbf{S5}}$(A)=$Cn_{\mathbf{S5}}$(C), then S∸A=S∸C. (*Extensionality*)

There are also two additional postulates similarly as in the *severe withdrawal* literature:

(C'7) If A∉$Cn_{\mathbf{S5}}$(∅), then S∸A⊆S∸(A∧C). (*Antitony*)

(C'8) If A∉S∸(A∧C), then S∸(A∧C)⊆S∸A. (*Conjunctive inclusion*)

Similarly as in Pagnucco [Pag96] and [RP99], we can obtain the following four postulates from CC'1-C'8.

(C'7c) If C∈S∸(A∧C), then S∸A⊆S∸(A∧C)

(C'8c) If C∈S∸(A∧C), then S∸(A∧C)⊆S∸A

(C'9) If A∉S∸C, then S∸C⊆S∸A

(C'10) If A∉$Cn_{\mathbf{S5}}$(∅) and A∈S∸C, then S∸A⊆S∸C

Proposition 24: Let ∸ be a severe withdrawal function over S. Then

1. Either S∸A⊆S∸C or S∸C⊆S∸A.

2. Either $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} (A \wedge C) = S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A$ or $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} (A \wedge C) = S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} C$.

3. If $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} (A \wedge C) \subseteq S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} C$, then $C \notin S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A$ or $A \in Cn_{S5}(\varnothing)$ or $C \in Cn_{S5}(\varnothing)$.

4. If $A \notin Cn_{S5}(\varnothing)$ and $C \notin Cn_{S5}(\varnothing)$, then either $A \notin S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} C$ or $C \notin S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A$.

A natural idea next is to explore the relationships between AGM contraction function $\div$ and severe withdrawal function $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$. Similarly as in [RP99], we define these operators from each other as in the following.

Definition 17 (Def $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ from $\div$): $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A = \begin{cases} \{C : C \in S \div (A \wedge C)\} & \text{if } \nvdash_{S5} A, \\ S & \text{otherwise.} \end{cases}$

Intuitively it means that in giving up A, "we should retain those beliefs that are always retained when given a choice between giving up A or another belief. That is, we retain those beliefs that are always retained when there is the possibility of removing either A or another sentence (or both)" [RP99].

Proposition 25: If $\div$ is an AGM contraction function, then $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ as obtained by (Def $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ from $\div$) is a *severe withdrawal* function which is revision equivalent to $\div$, and for all $A \in L$, $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A \subseteq S \div A$.

Definition 18 (Def $\div$ from $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$): $S \div A = \begin{cases} S \cap Cn_{S5}(S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A \cup \{\neg A\}) & \text{if } \nvdash_{S5} A, \\ S & \text{otherwise.} \end{cases}$

The interesting point is to guarantee the *recovery* postulate of $\div$ although $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ does not satisfy this in general.

Proposition 26: If $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ is a *severe withdrawal* function, then $\div$ as obtained by (Def $\div$ from $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$) is an AGM contraction function which is revision equivalent to $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$, and for all $A \in L$, $S \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} A \subseteq S \div A$.

Proposition 27 (relating two definitions):

(1) If we start with an AGM contraction function $\div$, turn it into a *severe withdrawal* function $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ by (Def $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ from $\div$) and turn the latter into another AGM contraction $\div'$ by (Def $\div$ from $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$), then $\div' = \div$.

(2) If we start with a *severe withdrawal* function $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$, turn it into an AGM contraction function $\div$ by (Def $\div$ from $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$) and turn the latter into another *severe withdrawal* function $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}'$ by (Def $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ from $\div$), then $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}} = \mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}'$.

This result shows that (Def $\div$ from $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$) and (Def $\mathbin{\overset{\cdot}{\underset{\cdot}{\sim}}}$ from $\div$) induce a 1-1 correspondence between (revision equivalent) AGM contraction functions and *severe withdrawal* functions. We will apply this property in the following *sphere system* and e*pistemic entrenchment relation* system.

5.2. Sphere system

In classical AGM belief revision literature, an interesting view of belief change is considering it in terms of possible worlds. A construction in this way specially focusing on AGM framework has been proposed by [Gro88]. [RP99] makes a review of Grove's work and define *severe withdrawal* (a contraction operator different from AGM contraction) from the sphere system. There, as Grove does, the current beliefs of an agent are characterized by the collection of possible worlds that are consistent with agent's beliefs. The remaining worlds which are inconsistent with agent's beliefs are grouped around the core collection in decreasing order of plausibility. This generates a system of spheres centered the set of possible worlds which are consistent with agent's beliefs. Then belief change can be expressed by the preference ordering over the possible worlds. Actually for classical belief change, *partial meet contraction* and *severe withdrawal* (it does not satisfy *recovery*) have been represented by the sphere system. There is a 1-1 correspondence relation between *partial meet contraction* (resp., *severe withdrawal*) and sphere system. Interestingly, although these two contraction operators are different, the corresponding revision operators defined via *Levi identity* in terms of sphere system are equivalent.

We are going to make an observation on revision of stable sets in a similar way in terms of possible worlds. Here every possible world is an **S5**-maximial consistent set. And the core set of possible worlds is defined by the set of worlds which are *consistent* with the current stable set S of an agent (it is reasonable to represent an ideal rational full introspective agent's beliefs in this way since all **S5** theorems are in any stable sets). This core set is presented by the minimal sphere in the sphere system. It is clear that those possible worlds are maximal consistent sets containing the stable set S. And the remaining worlds which are *inconsistent* with the stable set S will be presented by nested spheres centered on the minimal sphere. Formally, we denote the core set of possible worlds which is consistent with S by [S] and the set of all possible worlds by W. So a system $\mathcal{S}$ of spheres centered on [S] is a set of nested spheres in which the innermost is [S] and the outermost sphere is W. Then similarly we can define *partial meet contraction* function and *severe withdrawal* function for stable sets in terms of such kind of possible worlds. If the consistent stable set S is maximal, then the sphere of core set is just a point, that is, only one possible world which is equal to S is concerned.

Definition 19 ($C_{\mathcal{S}}$(A)): Let $\mathcal{S}$ be a sphere system and A a formula over L. The smallest sphere in $\mathcal{S}$ intersecting [A] (abbreviation of [{A}]) is denoted by $C_{\mathcal{S}}(A)$, that is, there is a sphere $U\in\mathcal{S}$ such that $U\cap[A]\neq\varnothing$, and $V\cap[A]\neq\varnothing$ implies $U\subseteq V$ for all $V\in\mathcal{S}$.

Definition 20 (Def ÷ from $\mathcal{S}$): Let S be a stable set and A a formula in L. $\mathcal{S}$ is a sphere system generated from S and ÷ is a AGM contraction (*partial meet*) on S. Define $S\div A=\mathcal{Th}([S]\cup f_{\mathcal{S}}(\neg A))$. Where $f_{\mathcal{S}}(A)=C_{\mathcal{S}}(A)\cap[A]$ and $\mathcal{Th}$ operator is different from classical Th operator. We will define it later in terms of Kripke models and some additional conditions.

Definition 21 (Def $\dot{-}$ from $\mathcal{S}$): Let S be a stable set and A a formula in L. $\mathcal{S}$ is a sphere system generated from S and $\dot{-}$ is a Rott contraction (*severe withdrawal*) on S. Define $S\dot{-}A=\mathcal{Th}(C_{\mathcal{S}}(\neg A))$.

We can see these two definitions intuitively in the following diagrams:

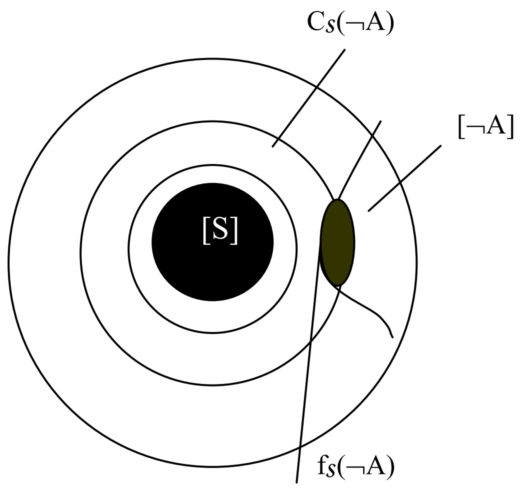

Figure 1: [S÷A] shaded

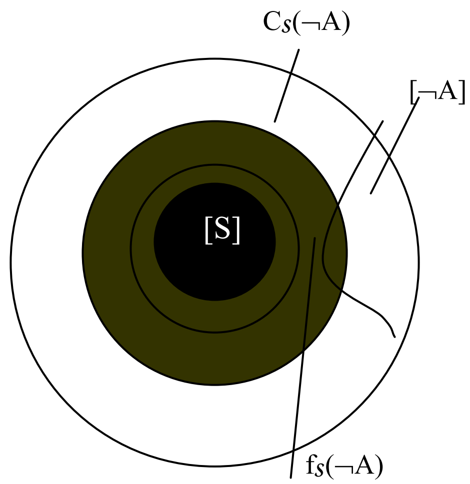

Figure 2: [S$\dot{-}$A] shaded

Before exploring the relationship between contraction operators (and so revision operators) and the sphere system, we need to conclude principles of the sphere system first. Different from [Gro88], before stating the principles, we are going to construct a canonical Kripke model whose domain consists of all **S5** maximal consistent sets.

Definition 22: Let W be the set of all possible worlds each of which is an **S5** maximal consistent set over the language L. Let S be a collection of subsets of W. Construct a Kripke model M based on W such that M=<W, R, v>. For every A in L, define a relation $R\subseteq W\times W$ by $(w, u)\in R$ iff $\{A: \mathbf{B}A\in w\}\subseteq u$. And for any w in W and propositional variable p, we define $v_w(p)=\mathbf{t}$ if and only if $p\in w$.

Next as in classical modal logic, the semantic entailment operator needs to be defined.

Definition 23: For every w in the Kripke model M=<W, R, v> and any formulae A in L,

If A is a propositional variable p, then $(M, w)\vDash p$ iff $v_w(p)=\mathbf{t}$.

If A=¬C, then (M, w)⊨A iff M, w⊭C.

If A=C→D, then (M, w)⊨A iff w⊨¬C or w⊨D.

If A=**B**C, then (M, w)⊨A iff for all w'∈W and wRw', then w'⊨C.

Now we are ready to prove the fundamental theorem for canonical models.

Proposition 28: For every w of the canonical model M and an arbitrary formula A in L, (M, w)⊨A if and only if A∈w.

Definition 24: Let R⊆W×W.

1. If for every w∈W, wRw, then R is called *reflexive*.
2. If for all w, u∈W, such that if wRu then uRw, then R is called *symmetric*.
3. If for all w, u, v∈W, such that if wRu and uRv then wRv, then R is called *transitive*.
4. If R is *reflexive*, *symmetric* and *transitive*, then R is called an *equivalence* relation.
5. If for all w, u, v∈W, such that if wRu and wRv then uRv, then R is called *Euclidean*.
6. If R=W×W, then R is called *universal*.

Proposition 29: R is an *equivalence* relation if only if it is *reflexive*, *transitive* and *Euclidean*.

Proposition 30: Let M=<W, R, v> be a canonical model constructed from W as in the above. Then R is an *equivalence* relation.

Definition 25: For any subset U⊆W, Th(U)={A: M, w⊨A for all w∈U}.

Remark: We attempt to obtain an operator Th from subsets of possible worlds to theories of L which is anti-monotonic. For the extreme cases, we have Th(W)=**S5** and Th(∅)=L. Classical corresponding operator Th is defined on models which may be non-monotonic.

Proposition 31: For any subset U⊆W, Th(U)=∩U.

Proposition 32: For every U⊆W, Th(U) is an **S5** theory. And If U⊆V for any U, V⊆W, then Th(V)⊆Th(U).

Proposition 33: For every U⊆W and A∈L, Cn(Th(U)∪{A})=Th(U∩[A]).

Proposition 34: For every U∈W and A∈L, Cm(Th(U)∪{A})=Cm(Th(U∩[A])).

Definition 26: We call S a system of spheres, centered on [S]⊆W (S is an arbitrary stable set), if it satisfies the following conditions:

(S1) S is totally ordered by⊆; that is, if U, V∈S, then U⊆V or V⊆U.

(S2) [S] is the ⊆-minimum of S.

(S3) W is the ⊆-maximum of S.

(S4) If A∈L and $\nvdash_{S5}$¬A, then there is a smallest sphere C_S(A) in S intersecting [A].

For $C_{\mathcal{S}}$, we need to consider the special case in which ¬A is an **S5** theorem. If $\vdash_{S5}$¬A, we define $C_{\mathcal{S}}$(A)=[S].

Next we are going to show that the contraction operators ÷ and ∻ defined from ***S*** which satisfies ***S***1- ***S***4 satisfy C1-C8 and C'1-C'4 plus C'6-C'8 respectively.

Theorem 3: Let S be a stable set and A a formula over L. ***S*** is a system of spheres satisfying ***S***1-***S***4. ÷ is a contraction operator on S with A defined from ***S*** by definition 4.4. Then operator ÷ satisfies *closure, success, inclusion, vacuity, recovery*, *extensionality,* and actually also *conjunctive overlap* (C7) and *conjunctive inclusion* (C8).

Theorem 4: Let S be a stable set and A a formula over L. ***S*** is a system of spheres satisfying ***S***1-***S***4. ∻ is a contraction operator on S with A defined from ***S*** by definition 4.5. Then ∻ satisfies *closure, success, inclusion, vacuity, extensionality, antitony (C'7)* and *conjunctive inclusion (C'8)*). But in general ∻ does not satisfy *recovery*.

Proposition 35: The revision operator R(÷) which is defined from ÷ and R(∻) which is defined from ∻ are equivalent, that is ((S÷¬**B**A)∩**B**S)+A=((S∻¬**B**A)∩**B**S)+A for any stable set S and formula A.

Theorem 5: Let * be the revision operator defined from ÷ or ∻ by Levi identity and ÷ and ∻ are defined from ***S***. Then * satisfies M1-M6 if ***S*** satisfies ***S***1-***S***4.

Like *partial meet contraction*, we hope to have the converse result, that is, to find a sphere system ***S*** that is defined from contraction operator ÷ or ∻ which satisfies C1-C8 (or C'1-C'8, and ∻ does not satisfy *recovery*), and ***S*** satisfies ***S***1-***S***4. We need to determine each sphere X_A (the minimal sphere intersecting [¬A] first. Then $\mathcal{S}$ can be defined via $\mathcal{S}=\{X_A: A\in L\}\cup\{W\}$ since every stable set S here is consistent.

Definition 27 (Def $\mathcal{S}$ from ÷): $X_A=\begin{cases}\bigcup\{[S\div C]: [C]\subseteq[A]\} & \text{if } \nvdash_{S5} A\\ [S] & \text{otherwise}\end{cases}$

Definition 28 (Def $\mathcal{S}$ from ∸): $X_A=[S\dot{-}A]$

And now we are going to consider the relationship between S obtained (by Def $\mathcal{S}$ from ÷) and S'(by Def $\mathcal{S}$ from ∸).

Proposition 36: The sphere system $\mathcal{S}$ and $\mathcal{S}'$ obtained via (Def $\mathcal{S}$ from ÷) and (Def $\mathcal{S}$ from ∸) respectively are equivalent.

Theorem 6: $\mathcal{S}$ (Def $\mathcal{S}$ from ÷ or Def $\mathcal{S}$ from ∸) satisfies $\mathcal{S}$1-$\mathcal{S}$4 where ÷ satisfies C1-C8 (and ∸ satisfies C'1-C'4 and C'6-C'8).

5.3 Epistemic entrenchment

In this subsection, we are going to construct a more intuitive mechanism like sphere system for the revision process: First we do contraction on S by ¬**B**A according to the definition of contraction like [RP99] from *epistemic entrenchment* relation similar as in [GM88], then do expansion with the help of logic **S5**.

If new information A is applied to revise S, given that {A} is *PI-consistent*, we need to consider the *PI-consistency* of S∪{A} since we know all **S5** theorems belong to every stable set. If S∪{A} is *PI-consistent*, then ¬**B**A cannot be in S. Since stable set is complete with respect to belief literals, then **B**A should be in S and A too by **Axiom schema T** and logical consequence. Hence we need not to discard any information to make room for A and get the new stable set S' just as S∪{A}=S. But if it is not *PI-consistent*, that is, ¬**B**A is in S (¬A or **B**¬A may be in S too, but either of them is the case then ¬**B**A can be guaranteed there since it can be derived by ¬A or **B**¬A in **S5**). We need to do contraction on S with ¬**B**A first similarly as in AGM revision processes. Like *sphere system*, next we are going to construct an epistemic entrenchment relation ≤ over all the formulae in L. Similar as in [GM88], there are five principles of ≤.

(E1) If A≤B and B≤C, then A≤C. (transitivity)

(E2) If A $\vdash_{S5}$ B, then A≤B. (dominance)

(E3) A≤A∧B or B≤A∧B. (conjunctiveness)

(E4) If the belief set S which is a stable set, then A∉S if and only if A≤B for all B. (minimality)

(E5) B≤A for all B, then $\vdash_{S5}$ A. (maximality)

Proposition 37: A≤B or B≤A for every A, B∈L.

The above observation shows that ≤ is a total order. Now we define its strict part A<B as A≤B and B≰A. We take the definition of contraction from entrenchment similarly as in [RP99] to connect entrenchment with contraction.

Definition 29 (Def ∸ from ≤)

$S\dot{-}A=\begin{cases}S\cap\{B: A<B\} & \text{if } A\in S \text{ and } \nvdash_{S5}A\\ S & \text{otherwise}\end{cases}$

Here S is a stable set and so it is closed under Cm and Cn_{S5}.

Similarly via *Levi identity* (variant version), we define a revision operator * on stable set S with A as: S*A=((S∸¬**B**A)∩**B**S)+A. It means intuitively that we first do contraction on S by ¬**B**A according to the definition ∸ from ≤ and restrict the result formulae in **B**S, and then do expansion on the contracted theory with A. Here T+A=Cm(T∪{A}) like before for any T and A. There is a question that why we first give up ¬**B**A but not ¬A. We will explain it in the process of proving consistency of S'. And someone may also ask that can it be guaranteed that if S is consistent and A is *PI-consistent*, then S' is also consistent? The answer seems positive like in *partial meet contraction*.

Proposition 38: If a stable set S is consistent and A is *PI-consistent*, then S'=((S∸¬**B**A)∩**B**S)+A is also consistent, where ∸ is defined from **≤.**

More generally we hope the revision operator * obtained from ≤ satisfies M1-M6 and conversely the entrenchment relation ≤ can be constructed from ∸ satisfying all five principles.

Theorem 7: Let belief revision operator * on stable sets be defined from ∸ via Levi identity, that is, S*A=((S∸¬**B**A)∩**B**S)+A. Then * satisfies M1-M6 if the epistemic entrenchment relation ≤ upon which the contraction operator ∸ is defined satisfies E1-E5.

Like in sphere system, we hope to have the converse result, that is, we want to construct some orderings from contraction operators and it is expected that those orderings satisfy E1-E5. Here we are going to observe orderings defined from AGM contraction and *severe withdrawal*. For any A, C∈L and a stable set S, we define the two orderings as in the following.

Definition 30: (Def ≤ from ∸) A≤C if and only if A∉S∸C or $\vdash_{\mathbf{S5}}$C.

Definition 31: (Def ≤ from ÷) A≤C if and only if A∉S÷(A∧C) or $\vdash_{\mathbf{S5}}$(A∧C).

We also define their strict part, such as (Def < from ÷): A<C if and only if C∈S÷(A∧C) and A∉S÷(A∧C). Intuitively this means C is retained but A is given up, showing that C is strictly more epistemic entrenchment than A.

Proposition 39: The epistemic entrenchment relation ≤ obtained by (Def ≤ from ∸) and ≤' obtained by (Def ≤ from ÷) are equivalent.

Theorem 8: Relation ≤ obtained by (Def ≤ from ∸) or equivalently by (Def ≤ from ÷) satisfies E1-E5.

Like in [RP99], we can also observe the relationships between *sphere system* and *epistemic entrenchment*. They may be not trivial. But here the central theme is to do revision on a stable set to obtain a new stable set rationally. We have only got the intermediate theory S'. The more important thing is to find a desired stable set which is expanded from S'.

6. Obtain a desired stable set from its corresponding intermediate theory

From the above different kinds of revision, we can obtain a *PI-consistent* intermediate theory S' from a stable set S, which is closed under Cm. But we know it cannot be taken as the result of revision from S for it may be not stable. And now we need to obtain just one stable set T=S#A containing S' such that # satisfies all the fundamental revision postulates for stable sets.

We know by [MT93] that every belief theory U in L that is consistent with **S5** (actually U is *PI-consistent*), then there is a stable and consistent theory T such that U⊆T.

Definition 32: Let W be a non-empty set and let R⊆W×W. If W'⊆W is a minimal non-empty subset of W such that for every w, w'∈W if (w, w')∈R then either w, w'∈W' or w, w'∈W\W', then the relation {(w, w'): w, w'∈W', (w, w')∈R is called a *component* of R.

Definition 33: If M=<W, R, v> is a Kripke structure and R|W' is a component of the accessibility relation R, then <W', R|W', v|W'> is a *component* of M.

Proposition 40: Let M be a Kripke structure and let $\{M_i\}i\in\omega$ be the family of all components of M. Then for every formula A∈L, M⊨A if only if for every $i\in\omega$, M_i⊨A.

Proposition 41 ([MT93]): Let M be a Kripke model with a universal accessibility relation. Then Th(M) is stable, where Th(M)={A: M⊨A}.

Corollary 2 ([MT93]): Let U⊆L be consistent with **S5**. Then there is a stable and consistent theory T such that U⊆T.

Now we are going to consider the intermediate theory S' which is derived from S by partial meet revision with A. In theorem 2, we have proved revision operator * satisfies *consistency*. So for any A which is *PI-consistent*, we can obtain a *PI-consistent* theory S' from stable set S revised by A. By corollary 2, we can always find a consistent stable set T such that $S' \subseteq T$. The problem here is that T may be not unique, that is, there may be more than one such kind of consistent stable sets containing S'. Then the postulate S5 may not be satisfied. So we need to construct a mechanism to obtain a definite consistent stable set T which contains S'.

Let $Z=Exp(S')=\{T|S' \subseteq T$ and T is stable and consistent$\}$. It is the set of all possible consistent stable sets containing S'. By [MT93], we know every T is maximal theory consistent with **S5** and every two of them are not included each other. Since we know by [Kon88, 93], [MT93] that every stable set is determined by its objective part, it is sufficient to focus on all their objective formulae to make a selection. And only on those parts can we compare the size of stable sets.

It is reasonable to consider the problem with the help of *information value*. For conservatism, the objective part of a stable set with minimal information value will be selected; but for adventurers, it is maximal. We are going to focus on the information value of objective sentences for the corresponding alternative stable sets.

Definition 34: A preference relation < is constructed over L_0. It is expected to be strictly linear. Like [GM88], there are three principles in the following:

1. If A<C and C<D, then A<D. (transitivity)

2. If $A \vdash C$ but $C \nvdash A$, then A<C. (dominance)

3. If $A \nvdash C$ and $C \nvdash A$, then $A<A \wedge C$ or $C<A \wedge C$. (conjunctiveness)

From those three principles we can conclude that for any two formulae A and C with $Cn(A) \neq Cn(C)$ are comparable in terms of <.

Proposition 42: For arbitrary A, $C \in L_0$, if $Cn(A) \neq Cn(C)$, then A<C or C<A.

With this preference relation we can compare the information value of any two un-equivalent objective formulae.

Definition 35: For objective formulae A, C, say information value of A is smaller than C (denoted by IV(A)<IV(C)) if C<A.

If Cn(A)=Cn(C), then A and C are not comparable in terms of <, they are equal with respect to information value. But actually we are interested in objective formulae which are not equivalent. A representative will be selected to represent the respective equivalent class. Objective theories will be treated same. Then it is clear that for an objective theory X after filtration, there is only one formula with *greatest information value* and one formula with *smallest information value*, denoted by Gre(X) and Sma(X); one second great and one second small, denoted by $Gre^1(X)$ and $Sma^1(X)$; …; and so on.

Definition 36: For any two objective theories X and Y, say information value of X is smaller than Y (denoted by IV(X)<IV(Y)) if one of the following conditions are satisfied:

1. Gre(X)<Gre(Y);
2. Gre(X)=Gre(Y), $Gre^1(X)<Gre^1(Y)$;
3. Gre(X)=Gre(Y), …, $Gre^{n-1}(X)=Gre^{n-1}(Y)$, $Gre^n(X)<Gre^n(Y)$ $(n \geq 2)$;
4. Gre(X)=Gre(Y), …, $Sma(X)=Gre^k(Y)$ (for some $k \in N$ and $Gre^k(Y)Sma(Y)$).

Proposition 43: every two un-equivalent objective theories are comparable in terms of <.

If two objective theories are equivalent, their information values are taken as equal. And actually their corresponding stable expansions are equal too.

Now we go back to the objective parts of stable sets each of which contains S'. They are pairwise un-equivalent since their corresponding stable sets are different. By proposition 43, we can select the objective theory which has the smallest information value in terms of <, and take its corresponding stable set as the desired stable set. Let f be a function on Z satisfying f(Z)=T $(IV(T \cap L_0)<IV(T' \cap L_0)$ for any $T' \in Z)$. Then define # from stable sets and new information to stable sets as S#A=f(Z).

Theorem 9: Revision operator # defined from f and * satisfies S1-S5.

7. Conclusion and further work

After considering difficulties and critical points in revising stable sets in this article, we apply **S5** non-implying subsets of a stable theory to construct "partial meet contraction" similar as in AGM theory; revision from a stable theory to an intermediate theory is presented, and a representation theorem between "partial meet contraction from stable sets to intermediate theories and postulates of contraction has been proved; it is also showed there that the revision operator from stable sets to intermediate theories satisfies some postulates we want; we put forward some notions of "positive introspective expansion" and variant version of "Levi Identity" to guarantee the revised intermediate theory to have at least one stable expansion. And we also constructed a "sphere system" and "epistemic entrenchment ordering" for revision from stable sets to intermediate theories similar as in classical belief revision theory; some representation theorems between them and contraction operators have been proved respectively; and it is also showed that the respective revision operators defined from those two semantical representations via variant version of "Levi Identity" satisfy all rational postulates we want. Then a general selection method is provided to select a desired stable set within stable expansions with the help of *information value*.

Revision operator # on a stable set we have done is functional, that is, for every stable set and new information, there is only one stable set obtained after the revision. However, in Lindström & Robinowicz (1989, 1990, 1992, 1994) [LR89], [LR90], [LR92], [RL94], a more general revision method was put forward which is called non-deterministic revision. There belief revision was treated as a relation GR_AH between theories (belief sets) rather than as a function on theories. For theory G and new information A, there may be more than one reasonable theories obtained from revising G with A. Thus GR_AH means that H is one of those reasonable revisions of the theory G with the new information A. In section 6, we have constructed a selection function attempting to obtain just one stable set among stable sets expanded from the intermediate theory. It seems reasonable that we take all those stable expansions of the intermediate theory as possible revisions. Then we can omit the selection work done before.

Furthermore, there is an assumption committed when we do revision work on stable sets, that is, every new information used for revision is expected to be *PI*-consistent; if it is *PI*-inconsistent then the revised theory would be inconsistent whatever the original stable set is. The reasons for inconsistency are *success* postulate and *stability* property of the outcome theories. Since the agents we are studying are fully introspective, it is reasonable to expect that the *stability* property holds in every revised theory. And from the point of view of *priority of new information*, postulate *success* has no problem either and it is actually widely accepted in belief revision literature. However some *PI*-inconsistent formula such as $p\wedge\neg\mathbf{B}p$ (p is a propositional variable) is consistent in classical modal logic. Then by postulate *consistency* in AGM tradition, consistent outcomes are expected since the new information is consistent. But it is obvious that we cannot obtain a consistent stable set which contains any form of $p\wedge\neg\mathbf{B}p$. So there seems no way to revise stable sets with *PI*-inconsistent new information if *classical consistency*, *stability* and *success* are satisfied. But in everyday life, we often announce *PI*-inconsistent propositions such as "it is raining, but you don't believe it" without the pain of inconsistency. Those announcements are treated normal in communication and they are actually used for information update and revision. It is then necessary for us to make a study on such a subject in some future.

Appendix: details of selected proofs

Proposition 11: If K is inconsistent, that is, $K=L_0$. Take $S=L$. It is an inconsistent stable theory expanded from K. It is clear that $S\cap L_0=L_0=K$.

Now consider the consistent case. Take $K'=K\cup\{\mathbf{B}A: A\in K\}\cup\{\neg\mathbf{B}A\in S: A\notin K \text{ and } A\in L_0\}$. Let $I=\mathbf{S5}+K'$. It is easy to show that I is consistent. For if it is not, we may consider two cases: one is that there exists an objective formula $A\in K$ such that $\neg A$ or $\neg\mathbf{B}A$ is an **S5** theorem. That means A is $\perp$, against the consistency of K. The other is that there exists an objective formula $A\notin K$ such that $\mathbf{B}A$ is a theorem of **S5**. Then A is also an **S5** theorem. For A is an objective formula, it must be a tautology in L_0, against $A\notin K$. Now list all formulae in L as $\psi_1, \ldots, \psi_i, \ldots$ Define the infinite sequence of sets $J_0, \ldots, J_i, \ldots$ as follows:

$J_0=I$

…

$$J_{i+1}=\begin{cases} Cn(J_i\cup\{\psi_{i+1}\}) \text{ if } \psi_{i+1} \text{ is consistent with } J_i \\ J_i \qquad\qquad\qquad \text{otherwise} \end{cases}$$

…

Let J=Cn($\cup$J$_i$). It is clear that J is consistent and complete **S5** theory.
Now take M={A: **B**A$\in$J}. We can prove that M is a consistent **S5** theory and actually M is a stable set.

Suppose that $\bot\in$M. Then **B**$\bot\in$J. But all **S5** theorems is contained in J, and by axiom **T** (**B**$\bot\to\bot$) we get $\bot$ is in J, contradicting that J is consistent. It is easy to see that all **S5** theorems are in M since all **S5** theorems are in J. It is not difficult to check that M is closed under classical logical consequence. Assume A and A$\to$C are in M. Then **B**A and **B**(A$\to$C) are in J. By Axiom **K** and MP we have **B**C in J. That means C is in M. Assume that A, C are in M. Then **B**A and **B**C are in J. We know **B**A$\wedge$**B**C$\to$**B**(A$\wedge$C) is an **S5** theorem, so **B**(A$\wedge$C) is also in J. Hence A$\wedge$C is in M.

Now we check that M is closed under A1 and A2. First we consider A1. Assume that **B**A$\in$M. Then **BB**A$\in$J. **BB**A$\to$**B**A is an **S5** theorem, then **B**A$\in$J, so A$\in$M. Assume that A$\in$M. Then **B**A$\in$J and by Axiom schema **4** we have **BB**A$\in$J. Hence **B**A$\in$M. Secondly we check A2. Assume that A$\notin$M. Then **B**A$\notin$J. **B**A$\leftrightarrow\neg$**B**$\neg$**B**A is an **S5** theorem, then $\neg$**B**$\neg$**B**A$\notin$J. Since J is a complete theory, we have **B**$\neg$**B**A$\in$J, hence $\neg$**B**A$\in$M, as required. Assume now that $\neg$**B**A$\in$M. Assume by contradiction that A$\in$M. Then we have **B**A$\in$J and by Axiom schema **4, BB**A$\in$J. Therefore we can conclude **B**A$\in$M by the construction of M, against the consistency of M.

So we have constructed an M that is a stable set. Finally we need to show that K$\subseteq$M and M$\cap$L$_0$=K.

First consider an arbitrary objective formula A from K. It is clear that A is in J. So **B**A is in J by the construction, and then A is in M, as required. Next we prove M$\cap$L$_0$=K. Since K$\subseteq$M and K$\subseteq$L$_0$, it is obvious that K$\subseteq$M$\cap$L$_0$. For the converse part, suppose A is a Boolean formula satisfying A$\notin$K. Then $\neg$**B**A$\in$J. But **B**A$\leftrightarrow\neg$**B**$\neg$**B**A is an **S5** theorem, then **B**$\neg$**B**A$\in$J, hence $\neg$**B**A$\in$M. But M is saturated, then A$\notin$M. Therefore M$\cap$L$_0\subseteq$K. This completes the proof. ┤

Proposition 23: It is trivial to prove "if" part. We only prove from "left to right". Suppose $\Sigma\vdash_N$ A. Then there is a finite proof sequence A$_1$, …, A$_n$=A such that for every A$_i$ (1≤i≤n), $\Sigma\vdash_N$ A$_i$. We prove by induction that for every 1≤i≤n, $\Sigma\vdash_{\mathbf{S5}}$ A$_i$. For base cases: if A$_i$ is an **S5** theorem or A$_i\in\Sigma$, then it is clear that $\Sigma\vdash_{\mathbf{S5}}$ A$_i$. For inductive cases: If A$_i$ is obtained from A$_k$ and A$_j$=A$_k\to$A$_i$ (k, j<i) by MP, then by induction hypothesis it follows that $\Sigma\vdash_{\mathbf{S5}}$ A$_k$ and $\Sigma\vdash_{\mathbf{S5}}$ A$_j$, and so we have $\Sigma\vdash_{\mathbf{S5}}$ A$_i$, as required. If A$_i$ is obtained from A$_l$ (l<i) by RN, then by induction hypothesis it follows that $\Sigma\vdash_{\mathbf{S5}}$ A$_l$. If A$_l$ is a belief atom **B**C, then we have $\Sigma\vdash_{\mathbf{S5}}$ **BB**C with the help of **Axiom schema 4**, that is, $\Sigma\vdash_{\mathbf{S5}}$ A$_i$. If A$_l$ is obtained from A$_c$ and A$_d$=A$_c\to$A$_l$ (c, d<l) by MP, then we have $\Sigma\vdash_N$ A$_c$ and $\Sigma\vdash_N$ A$_d$, and so $\Sigma\vdash_N$ **B**A$_c$ and $\Sigma\vdash_N$ **B**A$_d$. And it is clear that the proof lengths for **B**A$_c$ and **B**A$_d$ are smaller than the proof length for A$_i$. Then by induction hypothesis again that $\Sigma\vdash_{\mathbf{S5}}$ **B**A$_c$ and $\Sigma\vdash_{\mathbf{S5}}$ **B**A$_d$. We know **B**A$_d$=**B**(A$_c\to$A$_l$). It follows by **Axiom schema K** that $\Sigma\vdash_{\mathbf{S5}}$ **B**A$_c\to$**B**A$_l$. Then with the help of $\Sigma\vdash_{\mathbf{S5}}$ **B**A$_c$, we can conclude $\Sigma\vdash_{\mathbf{S5}}$ **B**A$_l$, that is $\Sigma\vdash_{\mathbf{S5}}$ A$_i$, as required. ┤

Theorem 2: We already have that if an operator ÷ for S is a partial meet contraction, then it satisfies *closure, success, inclusion, vacuity, recovery* and *extensionality* of basic contraction postulates. We know * is defined from ÷ and + via a different type of *Levi identity*. We may use the result to help this proof.

For M1(*PI-closure*), it follows directly from the definition of * that S*A=Cm(((S~$_\gamma\neg$**B**A)$\cap$**B**S)$\cup${A}).

For *consistency*, suppose A is *PI-consistent*, we need to show that Cm(((S~$_\gamma\neg$**B**A)$\cap$**B**S)$\cup${A}) is consistent. We know $\neg$**B**A will not be an **S5** theorem since A is *PI-consistent*. Then by *success* of ~$_\gamma$, we know $\neg$**B**A$\notin$S~$_\gamma\neg$**B**A. Now suppose Cm(((S~$_\gamma\neg$**B**A)$\cap$**B**S)$\cup${A}) is inconsistent. Then there exists a finite set Σ in (S~$_\gamma\neg$**B**A)$\cap$**B**S such that $\wedge\Sigma\wedge$A$\vdash_N\bot$. Since A is *PI-consistent*, it can be derived that $\Sigma\vdash_N$ $\neg$**B**A. We know every formula in Σ is a belief atom, that is, it has the form **B**C. Then by proposition 23, $\Sigma\vdash_N$ $\neg$**B**A is equivalent to $\Sigma\vdash_{\mathbf{S5}}$ $\neg$**B**A. It follows that $\neg$**B**A$\in$S~$_\gamma\neg$**B**A since Σ is a subset of S~$_\gamma\neg$**B**A and S~$_\gamma\neg$**B**A is closed under Cn$_{\mathbf{S5}}$ and Cn$_{\mathbf{S5}}$ is monotonic, against the result we have just obtained. Thus we can conclude that Cm(((S~$_\gamma\neg$**B**A)$\cap$**B**S)$\cup${A}) is consistent.

Next we check * satisfies *success*. It can be obtained directly from the definition of * that $A\in Cm(((S\sim_\gamma\neg\mathbf{B}A)\cap\mathbf{B}S)\cup\{A\})$. And actually we can derive a stronger version of *success*, that is, $\mathbf{B}^n A\in S*A$, where n is an arbitrary natural number.

And *inclusion* seems obvious since $S\sim_\gamma\neg\mathbf{B}A\subseteq S$ by *inclusion* of *partial meet contraction*. Now we check *vacuity*. Assume $\neg\mathbf{B}A\notin S$, we need to show $S*A=S$. If follows from $\neg\mathbf{B}A\notin S$ and S is a stable set (it is complete with respect to belief literals, that is, for every belief atom **B**A, **B**A or $\neg$**B**A is in the stable set) that $\mathbf{B}A\in S$ and so $A\in S$. It follows from $\neg\mathbf{B}A\notin S$ and *vacuity* of partial meet contraction that $S\sim_\gamma\neg\mathbf{B}A=S$. Therefore we get the result $Cm(((S\sim_\gamma\neg\mathbf{B}A)\cap\mathbf{B}S)\cup\{A\})=Cm(\mathbf{B}S\cup\{A\})=S$.

Last we show *extensionality*. Suppose Cm(A)=Cm(C). Then we have $Cn_{\mathbf{S5}}(\mathbf{B}A)=Cn_{\mathbf{S5}}(\mathbf{B}C)$. This means every **S5** model M is a model of **B**A is also a model of **B**C, and vice verse. So every **S5** model M is a model of $\neg$**B**A if only if it is a model of $\neg$**B**C. Then we get $Cn_{\mathbf{S5}}(\neg\mathbf{B}A)=Cn_{\mathbf{S5}}(\neg\mathbf{B}C)$. By the *extensionality* of contraction, it follows that $S\sim_\gamma\neg\mathbf{B}A=S\sim_\gamma\neg\mathbf{B}C$. It is sufficient to obtain $Cm(((S\sim_\gamma\neg\mathbf{B}A)\cap\mathbf{B}S)\cup\{A\})=Cm(((S\sim_\gamma\neg\mathbf{B}C)\cap\mathbf{B}S)\cup\{C\})$ since A and C are *PI-equivalent*. ⊣

Proposition 27: First we check (1). Suppose we have S÷A for any stable set S and $A\in L$. If A is an **S5** theory, then by (Def ∸ from ÷) we have S∸A=S and next by (Def ÷ from ∸) it follows that S÷'A=S=S÷A, as required. Now consider the interesting case that A is not an **S5** theorem. It follows by (Def ∸ from ÷) that $S∸A=\{C: C\in S÷(A\wedge C)\}$ and then by (Def ÷ from ∸) that $S÷'A=S\cap Cn_{\mathbf{S5}}(S∸A\cup\{\neg A\})=S\cap Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})$. We need to show that $S\cap Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})=S÷A$. Assume $D\in S\cap Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})$. It follows that $D\in S$ and $D\in Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})$. Then by *deduction* we have $\{C: C\in S÷(A\wedge C)\}\vdash_{S5}\neg A\to D$, that is, $A\vee D\in S÷(A\wedge(A\vee D))$ since $\{C: C\in S÷(A\wedge C)\}$ is closed under $Cn_{\mathbf{S5}}$. And we know that $A\wedge(A\vee D)$ is equivalent to A in classical logic, then by *extensionality* of ÷ that $S÷(A\wedge(A\vee D))=S÷A$. So we have $A\vee D\in S÷A$. And by recovery of ÷ we have $D\in Cn_{\mathbf{S5}}(S÷A\cup\{A\})$ since $D\in S$ and $S\subseteq Cn_{\mathbf{S5}}(S÷A\cup\{A\})$. Then it follows by *deduction* that $S÷A\vdash_{S5} A\to D$, that is, $\neg A\vee D\in S÷A$ since S÷A is closed under $Cn_{\mathbf{S5}}$. It follows from $A\vee D\in S÷A$, $\neg A\vee D\in S÷A$ and S÷A is closed under $Cn_{\mathbf{S5}}$ that $D\in S÷A$, as required. Next we show $S÷A\subseteq S\cap Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})$. Assume an arbitrary $D\in S÷A$. Then it is clear that $D\in S$. Suppose for reductio that $D\notin Cn_{\mathbf{S5}}(\{C: C\in S÷(A\wedge C)\}\cup\{\neg A\})$. Then $A\vee D\notin\{C: C\in S÷(A\wedge C)\}$ we have $A\vee D\notin S÷(A\wedge(A\vee D))$. Observe that $A\wedge(A\vee D)\leftrightarrow A$, by *extensionality* of ÷ we have $S÷(A\wedge(A\vee D))=S÷A$. So $A\vee D\notin S÷A$. This means $D\notin S÷A$ by the *closure* of ÷, a contradiction. Hence we obtained (1).

Next we prove (2). Suppose we have S∸A for any stable set S and $A\in L$. We only consider the main case that A is not an **S5** theory. Then it follows that $S÷A=S\cap Cn_{\mathbf{S5}}(S∸A\cup\{\neg A\})$ by (Def ÷ from ∸) and so $S∸'A=\{C: C\in S÷(A\wedge C)\}=\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}$ by (Def ∸ from ÷). We need to show $\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}=S∸A$.

From right to left: Assume $D\in S∸A$. If D is an **S5** theorem, then $D\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge D)\cup\{\neg(A\wedge D)\})$ and so $D\in\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}$, as required. If D is not an S5 theorem, then it follows by (C'7) that $S∸A\subseteq S∸(A\wedge D)$ and so $D\in S∸(A\wedge D)$. It means that $D\in\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}$, as required.

From left to right: Assume $D\in\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}$ for an arbitrary D. Then $D\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge D)\cup\{\neg(A\wedge D)\})$. It follows that $D\in S$ and $D\in Cn_{\mathbf{S5}}(S∸(A\wedge D)\cup\{\neg(A\wedge D)\})$. Then by *deduction* we can obtain that $S∸(A\wedge D)\vdash_{S5}\neg(A\wedge D)\to D$. It follows that $S∸(A\wedge D)\vdash_{S5}\neg A\to D$ and $S∸(A\wedge D)\vdash_{S5}\neg D\to D$, that is, $A\vee D\in S∸(A\wedge D)$ and $D\in S∸(A\wedge D)$ by the *closure* of ∸. It is clear that $A\notin S∸(A\wedge D)$ by *success* of ∸ since $A\wedge D$ is not an **S5** theorem. Then by (C'8) that $S∸(A\wedge D)\subseteq S∸A$. So we have $D\in S∸A$, as required. Now we can conclude that $\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}\subseteq S∸A$ from this proof. So we have $\{C: C\in S\cap Cn_{\mathbf{S5}}(S∸(A\wedge C)\cup\{\neg(A\wedge C)\})\}=S∸A$.⊣

Proposition 28: By induction on the structure of formula A. If A is a propositional variable p, assume $(M, w)\vDash p$, then by definition 22, $V_w(p)=\mathbf{t}$. This means $p\in w$ by the definition of v of the canonical model M. For the converse, assume $p\in w$. By the definition of v, $V_w(p)=\mathbf{t}$. This means $(M, w)\vDash p$ by definition 22. If A has the form $\neg C$. Suppose $(M, w)\vDash A$ (that is, $(M, w)\nvDash C$). By induction hypothesis, we have $C\notin w$. But w is maximal, so $\neg C\in w$, as required. Conversely, suppose $\neg C\in w$. It is trivial to get $(M, w)\vDash\neg C$. It is similar to prove the proposition holds in the case of that A has the form $C\rightarrow D$. The interesting case is that A has the form $\mathbf{B}C$. Suppose $(M, w)\nvDash\mathbf{B}C$. Then by definition 22, there is some w' in W such that wRw' and $(M, w')\nvDash C$. By induction hypothesis, it follows that $C\notin w'$. Since wRw'holds, then $\{A: \mathbf{B}A\in w\}\subseteq w'$. Suppose for the contrary $\mathbf{B}C\in w$. It follows by $\{A: \mathbf{B}A\in w\}\subseteq w'$ that $C\in w'$, contradicting $C\notin w'$. It follows by this contradiction that $\mathbf{B}C\notin w$. For the converse part, suppose $\mathbf{B}C\notin w$. Since w contains all **S5** theorems, then $\mathbf{B}C\notin \mathrm{Thm}(\mathbf{S5})$ (the set of **S5** theorems). It follows that C is not an **S5** theorem. This means $\{\neg C\}$ is consistent with **S5**. Then by Lindenbaum Lemma, we can expand $\mathrm{Thm}(\mathbf{S5})\cup\{\neg C\}$to obtain a maximal consistent set u. In order to obtain a particular maximal consistent set, let us construct an infinite sequence first as the following:

$$I_0=\mathrm{Thm}(\mathbf{S5})\cup\{\neg C\}$$

$$\ldots$$

$$I_{k+1}=\begin{cases} I_k\cup\{D\} & \text{if } \mathbf{B}D\in w \text{ and } D \text{ is consistent with } I_k \\ I_k\cup\{\neg D\} & \text{if } \mathbf{B}D\notin w \text{ and } \neg D \text{ is consistent with } I_k \\ I_k & \text{otherwise} \end{cases}$$

Where, D is an arbitrary formula in L. Let $u=\bigcup I$. It is clear that u is maximal and consistent. And it is clear that $C\notin u$. By induction hypothesis, $(M, u)\nvDash C$. It is easy to see that wRu. Suppose for an arbitrary $D\notin u$, we need to show $D\notin\{A: \mathbf{B}A\in w\}$. It follows from $D\notin u$ that $\neg D\in u$. By the construction of u, we have $\mathbf{B}D\notin w$ with $\neg D$ consistent with u or $\mathbf{B}\neg D\in w$ with $\neg D$ consistent with u. For the former, we can conclude that $D\notin\{A: \mathbf{B}A\in w\}$ since $\mathbf{B}D\notin w$, as required. And it follows that $\neg D \in\{A: \mathbf{B}A\in w\}$ by the latter case. Suppose for the contrary $D\in\{A: \mathbf{B}A\in w\}$. Then we have $\mathbf{B}D\in w$. It follows from $\mathbf{B}\neg D\in w$ and $\mathbf{B}D\in w$ that w is inconsistent, a contradiction. This means wRu holds. Thus $(M, w)\nvDash\mathbf{B}C$, that is, $(M, w) \vDash\neg\mathbf{B}C$, as required. ┤

Proposition 30: By proposition 29, it is sufficient to verify R is *reflexive*, *transitive* and *Euclidean*. First we verify that R is *reflexive*. For any $w\in W$, axiom schema $\mathbf{B}A\rightarrow A$ is in w since it is an **S5** theorem. Suppose for reductio that R is not reflexive. Then there is an A such that $\mathbf{B}A\in w$ and $A\notin w$ by definition 22. It follow from $\mathbf{B}A\rightarrow A\in w$ and $\mathbf{B}A\in w$ that $A\in w$, contradicting to supposition. Hence R is reflexive. Next we check R is *transitive*. Assume for arbitrary worlds w, u and v in W, and wRu, uRv hold. Suppose for reductio that $\neg(wRv)$. It follows by definition 22 that there is an A such that $\mathbf{B}A\in w$ and $A\notin v$. It then follows by wRu, uRv and definition 22 that $\mathbf{B}A\notin u$. It is clear that $\mathbf{B}A\rightarrow\mathbf{BB}A\in w$ since it is an **S5** axiom and so $\mathbf{BB}A\in w$. It follows by wRu that $\mathbf{B}A\in u$, contradicting to $\mathbf{B}A\notin u$. Hence we conclude from the contradiction that wRv, as required. Last for this proof, we verify R is *Euclidean*. Assume for arbitrary worlds w, u and v in W such that wRu, wRv hold. It follows by definition 22 that $\{A: \mathbf{B}A\in w\}\subseteq u$ and $\{A: \mathbf{B}A\in w\}\subseteq v$. Suppose for an arbitrary C such that $C\in\{A: \mathbf{B}A\in u\}$, we need to show $C\in v$. Suppose for reductio that $C\notin v$. It follows from $\{A: \mathbf{B}A\in w\}\subseteq v$ that $C\notin\{A: \mathbf{B}A\in w\}$ and so $\mathbf{B}C\notin w$. Then $\neg\mathbf{B}C\in w$ since w is maximal consistent. By the **S5** theorem $\neg\mathbf{B}C\rightarrow\mathbf{B}\neg\mathbf{B}C$ we can obtain $\mathbf{B}\neg\mathbf{B}C\in w$. It follows from $\{A: \mathbf{B}A\in w\}\subseteq u$ that $\neg\mathbf{B}C\in u$. But we know $C\in\{A: \mathbf{B}A\in u\}$, then $\mathbf{B}C\in u$. This shows u is inconsistent, contrary to the consistency of u. Hence we can conclude uRv, as required. ┤

Proposition 31: By definition 25, we have $A\in \mathrm{Th}(U)$ iff $M, w\vDash A$ for every $w\in U$, and then by fundamental theorem of canonical models, iff $A\in w$ for every $w\in U$, that is, $A\in\bigcap U$. ┤

Proposition 32: We know all **S5** theorems are in every w of M, so $M\vDash \mathrm{Thm}(\mathbf{S5})$. By proposition 31, $\mathcal{Th}(U)=\bigcap U$. It follows from $U\subseteq W$ that $\bigcap W\subseteq\bigcap U$. Since $\bigcap W=\mathrm{Thm}(\mathbf{S5})$, $\mathcal{Th}(U)=\bigcap U$ contains $\mathrm{Thm}(\mathbf{S5})$. Now we need only to check $\mathcal{Th}(U)$ is closed under $\mathrm{Cn}_{\mathbf{S5}}$. First we show it is closed under Cn. Suppose arbitrary formulae A and A→C are in $\mathcal{Th}$(U). By definition 25, we have (M, w)⊨A and (M, w)⊨A→C for every w∈U. But we know every w is a maximal consistent set, it is closed under Cn. Therefore (M, w)⊨C for every w∈U. It means that C is also in $\mathcal{Th}$(U). Last for this proof, we suppose U⊆V. By proposition 31, we have $\mathcal{Th}$(V)=∩V and $\mathcal{Th}$(U)=∩U. If follows by set theory and U⊆V that ∩V=∩U. Hence $\mathcal{Th}$(V)⊆$\mathcal{Th}$(U). ⊣

Proposition 33:

From left to right: Suppose for an arbitrary C∈Cn(Th(U)∪{A}). It follows by *deduction* that Th(U)⊢A→C. And A→C∈Th(U) since it is closed under Cn_{S5}. It means A→C∈w for every w of U. Consider the worlds in U∩[A]. They are the worlds satisfying that A→C and A are in each of them. So by MP we have C is in every world of U∩[A]. It suffices showing that C∈Th(U∩[A]).

From right to left: If U∩[A]=∅, then U⊂[¬A] and so ¬A∈$\mathcal{Th}$(U). It follows that $\mathcal{Th}$(U)∪{A} is inconsistent, that is, Cn($\mathcal{Th}$(U)∪{A})=L, as required. For the main case, that is, U∩[A]≠∅, suppose for an arbitrary C∉Cn($\mathcal{Th}$(U)∪{A}). It means $\mathcal{Th}$(U)∪{A}∪{¬C} is consistent. By Lindenbaum Lemma, there is a maximal consistent set w' such that $\mathcal{Th}$(U)∪{A}∪{¬C}⊆w'. We know $\mathcal{Th}$(U) is an **S5** theory, so is w'. It follows that w'∈W. By proposition 28, it can be derived that (M, w')⊨¬C, i.e., (M, w')⊭C. And we know (M, w')⊨$\mathcal{Th}$(U)∪{A}. It means w'∈U∩[A] for it satisfies all formulae in $\mathcal{Th}$(U) and A. Therefore we have (M, w')⊨$\mathcal{Th}$(U∩[A]) by the definition of $\mathcal{Th}$. It follows that $\mathcal{Th}$(U∩[A])⊬C by *soundness theorem* of classical logic, that is, C∉Cn($\mathcal{Th}$(U∩[A])). Note that $\mathcal{Th}$(U∩[A]) is closed under Cn by proposition 32, then C∉$\mathcal{Th}$(U∩[A]), as required. ⊣

Proposition 34: It can be obtained directly from proposition 33. We know Cm(Cn(Th(U) ∪{A}))=Cm(Th(U)∪{A}). It follows from Cn(Th(U)∪{A})=Th(U∩[A]) (proposition 33) that Cm(Cn(Th(U)∪{A}))=Cm(Th(U∩[A])), and so Cm(Th(U)∪{A})=Cm(Th(U∩[A])). ⊣

Theorem 3: We prove it satisfies all six fundamental postulates and two additional postulates one by one.

Closure can be obtained by the definition 20 and proposition 32 directly.

Now we are going to check ÷ satisfies *success*. Suppose A∉Cn_{S5}(∅), we need to show A∉$\mathcal{Th}$([S]∪f_S(¬A)). If A∉S, then it is obvious that A∉w for some w∈[S], and so ¬A∈w. We obtain A∉$\mathcal{Th}$([S]∪f_S(¬A)) since $\mathcal{Th}$([S]∪f_S(¬A))⊆$\mathcal{Th}$([S])=S. Consider the principle case A∈S. We know that A does not belong to any member of f_S(¬A). It follows from A∉w for every w∈f_S(¬A) that A∉u for some u in [S]∪f_S(¬A). Thus A∉$\mathcal{Th}$([S]∪f_S(¬A)).

Next we prove ÷ satisfies *inclusion*. It is obvious that [S]⊆[S]∪f_S(¬A). Then $\mathcal{Th}$([S]∪f_S(¬A)) ⊆$\mathcal{Th}$([S])=S, as required.

Proving *vacuity* is also easy. Suppose A∉S, we need to check S=$\mathcal{Th}$([S]∪f_S(¬A)). It follows from A∉S that there is some w∈[S] such that ¬A∈w. This means C_S(A)=[S]. Hence [S]∪f_S(¬A)=[S]∪([S]∩[¬A])=[S] and S=$\mathcal{Th}$([S]∪f_S(¬A)).

Next we are going to check *recovery*, that is, S⊆Cn_{S5}($\mathcal{Th}$([S]∪f_S(¬A))∪{A}). If A∉S, it is obvious to get this result by *vacuity*. Consider the principle case that A∈S. Notice that Cn_{S5}($\mathcal{Th}$([S]∪f_S(¬A))∪{A})=Cn_{S5}($\mathcal{Th}$(([S]∪f_S(¬A))∩[A])), it is sufficient to show ([S]∪f_S(¬A)) ∩[A]⊆[S]. Since A∈S, then we have [S]⊆[A]. And we know f_S(¬A)∩[A]=∅ since A and ¬A are inconsistent. It follows from ([S]∪f_S(¬A))∩[A]=([S]∩[A])∪(f_S(¬A)∩[A])=[S]∩[A] and [S]⊆[A] that ([S]∪f_S(¬A))∩[A]=[S], as required.

Next we prove *extensionality*. Assume Cn_{S5}(A)=Cn_{S5}(C), we need to show $\mathcal{Th}$([S]∪f_S(¬A)) =$\mathcal{Th}$([S]∪f_S(¬C)). It is sufficient to show f_S(¬A)=f_S(¬C). It follows from Cn_{S5}(A)=Cn_{S5}(C) that Cn_{S5}(¬A)=Cn_{S5}(¬C). Then we have [¬A]=[¬C] and so C_S(¬A)=C_S(¬C). Therefore C_S(¬A)∩[¬A] =C_S(¬C)∩[¬C], as required.

Now we are ready to check ÷ satisfies *conjunctive overlap* and *conjunctive inclusion*.

First for (C7), assume we have S÷A∩S÷C. It follows by (Def ÷ from $\mathcal{S}$) that S÷A∩S÷C =$\mathcal{Th}$([S]∪f_S(¬A))∩$\mathcal{Th}$([S]∪f_S(¬C)). We know f_S(¬A)=C_S(¬A)∩[¬A] and f_S(¬C)=C_S(¬C)∩[¬C] by the definition of f_S. So S÷A∩S÷C=$\mathcal{Th}$([S]∪(C_S(¬A)∩[¬A]))∩$\mathcal{Th}$([S]∪(C_S(¬C)∩[¬C])). Similarly we have S÷(A∧C)=$\mathcal{Th}$([S]∪f_S(¬(A∧C)))=$\mathcal{Th}$([S]∪(C_S(¬A∨¬C)∩[¬A∨¬C])). Now to verify $S\div A\cap S\div C\subseteq S\div(A\wedge C)$ is just to verify $C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]\subseteq(C_{\mathcal{S}}(\neg A)\cap[\neg A])\cup(C_{\mathcal{S}}(\neg C)\cap[\neg C])$. Suppose for an arbitrary $w\in C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]$. Then $w\in C_{\mathcal{S}}(\neg A\vee\neg C)$ and $w\in[\neg A\vee\neg C]=[\neg A]\cup[\neg C]$. Since $[\neg A\vee\neg C]\supseteq[\neg A]$ and $[\neg A\vee\neg C]\supseteq[\neg C]$, then it is clear that $C_{\mathcal{S}}(\neg A\vee\neg C)\subseteq C_{\mathcal{S}}(\neg A)$ and $C_{\mathcal{S}}(\neg A\vee\neg C)\subseteq C_{\mathcal{S}}(\neg C)$. It follows from $w\in C_{\mathcal{S}}(\neg A\vee\neg C)$ and $C_{\mathcal{S}}(\neg A\vee\neg C)\subseteq C_{\mathcal{S}}(\neg A)$ and $C_{\mathcal{S}}(\neg A\vee\neg C)\subseteq C_{\mathcal{S}}(\neg C)$ that $w\in C_{\mathcal{S}}(\neg A)$ and $w\in C_{\mathcal{S}}(\neg C)$. And we know $w\in[\neg A]$ or $w\in[\neg C]$. So $w\in C_{\mathcal{S}}(\neg A)\cap[\neg A]$ or $w\in C_{\mathcal{S}}(\neg C)\cap[\neg C]$, that is, $w\in(C_{\mathcal{S}}(\neg A)\cap[\neg A])\cup(C_{\mathcal{S}}(\neg C)\cap[\neg C])$, as require.

Last of this proof is to check (C8). Let $A\notin S\div(A\wedge C)$ and we need to show $S\div(A\wedge C)\subseteq S\div A$. It can be turn into to show that $\mathcal{Th}([S]\cup(C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]))\subseteq\mathcal{Th}([S]\cup(C_{\mathcal{S}}(\neg A)\cap[\neg A]))$ by (Def $\div$ from $\mathcal{S}$). If $A\notin S$ then $S\div A=S$ by *vacuity* (has been proved) and $S\div(A\wedge C)\subseteq S$ by *inclusion* (has been proved). So we obtain the desired result. Now we consider the main case that $A\in S$. It is necessary to show that $C_{\mathcal{S}}(\neg A)\cap[\neg A]\subseteq C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]$. Since $A\notin S\div(A\wedge C)$, then $A\notin\mathcal{Th}([S]\cup(C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]))$. It follows from $A\in S$ and $A\notin\mathcal{Th}([S]\cup(C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]))$ that $A\notin\mathcal{Th}(C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C])$. This means $C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]\cap[\neg A]\neq\varnothing$, and so $C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A]\neq\varnothing$. It follows that $C_{\mathcal{S}}(\neg A)\subseteq C_{\mathcal{S}}(\neg A\vee\neg C)$. And it is clear that $[\neg A]\subseteq[\neg A\vee\neg C]$, then $C_{\mathcal{S}}(\neg A\vee\neg C)\subseteq C_{\mathcal{S}}(\neg A)$, this means $C_{\mathcal{S}}(\neg A\vee\neg C)=C_{\mathcal{S}}(\neg A)$ so we can obtain $C_{\mathcal{S}}(\neg A)\cap[\neg A]\subseteq C_{\mathcal{S}}(\neg A\vee\neg C)\cap[\neg A\vee\neg C]$, as required. $\dashv$

Theorem 4: Similarly as in proving theorem 3, we can show $\dot{\div}$ satisfies the above five fundamental postulates. Take *vacuity* as an example. Suppose $A\notin S$ for an arbitrary A. Then there is some $w\in[S]$ such that $\neg A\in w$. This means $C_{\mathcal{S}}(\neg A)=[S]$. Hence $S=\mathcal{Th}(C_{\mathcal{S}}(\neg A))$.

Now we are going to check that $\dot{\div}$ does not satisfy *recovery* in general. By the definition of $\dot{\div}$ we have $Cn_{S5}((S\dot{\div}A)\cup\{A\})=Cn_{S5}(\mathcal{Th}(C_{\mathcal{S}}(\neg A))\cup\{A\})=Cn(\mathcal{Th}(C_{\mathcal{S}}(\neg A))\cup\{A\})$. It follows from proposition 33 that $Cn(\mathcal{Th}(C_{\mathcal{S}}(\neg A))\cup\{A\})=\mathcal{Th}(C_{\mathcal{S}}(\neg A)\cap[A]))$. Consider the case where $A\in S$. It is possible that $C_{\mathcal{S}}(\neg A)\cap[A]=C_{\mathcal{S}}(\neg A)\text{-}f_{\mathcal{S}}(\neg A)$ but $C_{\mathcal{S}}(\neg A)\text{-}f_{\mathcal{S}}(\neg A)\not\subseteq[S]$ (please see the following diagram).

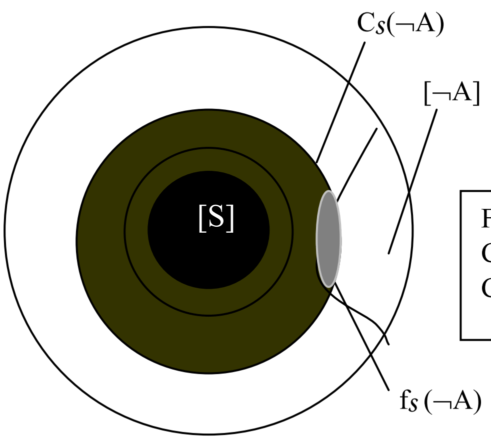

Figure 3: It is clear that $C_{\mathcal{S}}(\neg A)\cap[A]=C_{\mathcal{S}}(\neg A)\text{-}f_{\mathcal{S}}(\neg A)$ but $C_{\mathcal{S}}(\neg A)\text{-}f_{\mathcal{S}}(\neg A)\not\subseteq[S]$

This means $S\not\subseteq Cn_{S5}((S\dot{\div}A)\cup\{A\})$. Therefore in such cases *recovery* does not hold.

Next we check it satisfies (C'7) and (C'8). For *antitony*, let $\nvdash_{S5}A$ and we need to show $S\dot{\div}A\subseteq S\dot{\div}(A\wedge C)$. By (Def $\div$ from S) it is to show $Th(C_S(\neg A))\subseteq Th(C_S(\neg(A\wedge C)))$, that is, $C_S(\neg(A\wedge C))\subseteq C_S(\neg A)$, or equivalently, $C_S(\neg A\vee\neg C)\subseteq C_S(\neg A)$. It is clear that $\nvdash_{S5}A\wedge C$ since $\nvdash_{S5}A$. And we have $[\neg A]\subseteq[\neg A]\cup[\neg C]=[\neg A\vee\neg C]$, so clearly that $C_S(\neg A\vee\neg C)\subseteq C_S(\neg A)$.

For *conjunctive inclusion*, let $A\notin S\dot{\div}(A\wedge C)$ and we need to show $S\dot{\div}(A\wedge C)\subseteq S\dot{\div}A$. It follows by (Def $\div$ from S) that $S\dot{\div}(A\wedge C)=Th(C_S(\neg(A\wedge C)))$ and $S\dot{\div}A=Th(C_S(\neg A))$. Since $A\notin S\dot{\div}(A\wedge C)=Th(C_S(\neg(A\wedge C)))=Th(C_S(\neg A\vee\neg C))$, we have $C_S(\neg A\vee\neg C)\cap[\neg A]\neq\varnothing$. So it is clear that $C_S(\neg A)\subseteq C_S(\neg A\vee\neg C)$. By the property of Th we obtain that $Th(C_S(\neg A\vee\neg C))\subseteq Th(C_S(\neg A)$. This means $S\dot{\div}(A\wedge C)\subseteq S\dot{\div}A$. $\dashv$

Proposition 35: By the construction we know $((S\div\neg\mathbf{B}A)\cap\mathbf{B}S)+A=Cm((\mathcal{Th}([S]\cup f_S(\mathbf{B}A))\cap\mathbf{B}S)\cup\{A\})=Cm(\mathcal{Th}(([S]\cup f_S(\mathbf{B}A))\cap\mathbf{B}S)\cup\{\mathbf{B}A\})$. By proposition 34 we have $((S\dot{\div}\neg\mathbf{B}A)\cap\mathbf{B}S)+A=Cm(\mathfrak{Th}(([S]\cup f_{\mathcal{S}}(\mathbf{B}A)\cup[\mathbf{B}S])\cap[\mathbf{B}A]))$. It is clear that $Cm(\mathfrak{Th}(([S]\cup f_{\mathcal{S}}(\mathbf{B}A)\cup[\mathbf{B}S])\cap[\mathbf{B}A]))=Cm(\mathfrak{Th}(([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap[\mathbf{B}A]))=Cm(\mathfrak{Th}(f_{\mathcal{S}}(\mathbf{B}A))$. And similarly $((S\dot{\div}\neg\mathbf{B}A)\cap\mathbf{B}S)+A=Cm((\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S)\cup\{\mathbf{B}A\})=Cm(\mathfrak{Th}((C_{\mathcal{S}}(\mathbf{B}A)\cup[\mathbf{B}S])\cap[\mathbf{B}A]))$ $=Cm(\mathfrak{Th}((C_{\mathcal{S}}(\mathbf{B}A)\cup[S])\cap[\mathbf{B}A]))=Cm(\mathfrak{Th}(f_{\mathcal{S}}(\mathbf{B}A)))$. These two are exactly the same. ┤

Theorem 5: Suppose $\mathcal{S}$ satisfies $\mathcal{S}1$-$\mathcal{S}4$. By proposition 35, R(÷) and R($\dot{\div}$) are same, we only need to verify one of them satisfies M1-M6.

First we show * satisfies *PI-closure*. Since $S*A=Cm(\mathfrak{Th}(f_{\mathcal{S}}(\mathbf{B}A)))$, it is clear that S*A is closed under Cm.

Next we check *consistency*. Suppose A is *PI-consistent*. Then [A] is not empty and $f_{\mathcal{S}}(\mathbf{B}A)$ is not empty either. If $A\in S$, then $\mathfrak{Th}(f_{\mathcal{S}}(\mathbf{B}A))=S$. It is clear S is *PI-consistent* since we only consider consistent stable sets. For the main case that $A\notin S$, it is not difficult to verify that $(\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S)\cup\{A\}$ is *PI-consistent*. It suffices to show that $\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S\nvdash_N\neg\mathbf{B}A$ since $\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S$ is *PI-consistent*. Suppose for reductio that $\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S\vdash_N\neg\mathbf{B}A$. Then there is a finite $\Sigma\subseteq\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S$. It follows by proposition 23 that $\Sigma\vdash_{S5}\neg\mathbf{B}A$ since Σ consists of all belief atoms. Then we have $\neg\mathbf{B}A\in\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))$. But it is clear that $\neg\mathbf{B}A\notin\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))$ since S is *consistent* and $\neg\mathbf{B}A\notin w$ for every $w\in f_{\mathcal{S}}(\mathbf{B}A)$. We can conclude that $\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S\nvdash_N\neg\mathbf{B}A$ from this contradiction. Hence $Cm(\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S)\cup\{\mathbf{B}A\}$ is *PI-consistent*, so are $(\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S)\cup\{A\}$ and $Cm((\mathfrak{Th}([S]\cup f_{\mathcal{S}}(\mathbf{B}A))\cap\mathbf{B}S)\cup\{A\})$.

Now we verify *success*. Consider two cases depending whether $\neg\mathbf{B}A$ is an **S5** theorem or not. If $\vdash_{S5}\neg\mathbf{B}A$, then by definition 24 we have $C_{\mathcal{S}}(\mathbf{B}A)=[S]$. And $[\mathbf{B}A]=\varnothing$ since there is no world in W which is consistent with A. Then $f_{\mathcal{S}}(\mathbf{B}A)=C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]=\varnothing$. Hence $S*A=Cm(\mathfrak{Th}(\varnothing))=L$ and so $A\in S*A$, as required. If $\nvdash_{S5}\neg\mathbf{B}A$, then by $\mathcal{S}1$-$\mathcal{S}4$ we can find a $C_{\mathcal{S}}(\mathbf{B}A)$ containing [S] and $C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]\neq\varnothing$ such that $S*A=Cm(\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]))$. It is clear that A must be in every w of $f_{\mathcal{S}}(\mathbf{B}A)=C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]$ since $\mathbf{B}A\in w$ and $\mathbf{B}A\rightarrow A\in w$. It follows that $A\in\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A])$, as required.

Similarly we can prove that * satisfies *inclusion*. Suppose $C\in S*A=Cm(\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]))$. We know $S+A=Cm(\mathfrak{Th}([S])\cup\{A\})=Cm(\mathfrak{Th}([S]\cap[\mathbf{B}A]))$ by proposition 34. Since $[S]\subseteq C_{\mathcal{S}}(\mathbf{B}A)$ in all situations, it follows that $[S]\cap[\mathbf{B}A]\subseteq C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]$. Hence $\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A])\subseteq\mathfrak{Th}([S]\cap[\mathbf{B}A])$.

Next we verify *vacuity*. Suppose $\neg\mathbf{B}A\notin S$, we need to show S*A=S. It follows from $\neg\mathbf{B}A\notin S$ that $\mathbf{B}A\in S$ since S is stable. It follows that $[S]\subseteq[\mathbf{B}A]$. Then $C_{\mathcal{S}}(\mathbf{B}A)=[S]$ since [S] is the minimal sphere which intersects with $[\mathbf{B}A]$. It follows from $[S]\subseteq[\mathbf{B}A]$ and $C_{\mathcal{S}}(\mathbf{B}A)=[S]$ that $C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]=[S]$. Hence $S*A=\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A])=\mathfrak{Th}([S])=S$, as required.

The last part of this proof is to verify *extensionality*. Suppose Cm(A)=Cm(C). Then $[\mathbf{B}A]=[\mathbf{B}C]$ and so $C_{\mathcal{S}}(\mathbf{B}A)=C_{\mathcal{S}}(\mathbf{B}C)$. Hence $C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A]=C_{\mathcal{S}}(\mathbf{B}C)\cap[\mathbf{B}C]$ and so $\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}A)\cap[\mathbf{B}A])=\mathfrak{Th}(C_{\mathcal{S}}(\mathbf{B}C)\cap[\mathbf{B}C])$, i.e., S*A=S*C. ┤

Proposition 36: For every $A\in L$, if A is an **S5** theory, then it is obvious that $X_A=[S]=X_A'=[S\dot{\div}A]=[S]$. Consider the case that A is not an **S5** theorem, then $X_A=\bigcup\{[S\div C]: [C]\subseteq[A]\}$ via definition 27, and $X_A'=[S\dot{\div}A]$ via definition 28. If $A\notin S$, then $S\dot{\div}A=S$ and $S\div A=S$. So $X_A=\bigcup\{[S\div C]: [C]\subseteq[A]\}=[S]$ and $X_A'=[S\dot{\div}A]=[S]$, that is, $X_A=X_A'$. Now consider the main case that $A\in S$. Suppose for an arbitrary $w\in\bigcup\{[S\div C]: [C]\subseteq[A]\}$. We have $w\in[S\div C]$ for some C satisfying $[C]\subseteq[A]$. It is not difficult to conclude that $S\dot{\div}C\subseteq S\div C$ since $\dot{\div}$ gives up more information. Then we have $[S\div C]\subseteq[S\dot{\div}C]$. It follows that $w\in[S\dot{\div}C]$. And it is not difficult for us to have that $S\dot{\div}A\subseteq S\dot{\div}C$ for every C satisfying $[C]\subseteq[A]$. We can verify that $C\notin S\dot{\div}A$. Suppose not, then $C\in S\dot{\div}A$. Since $[C]\subseteq[A]$, it follows that $C\rightarrow A\in S\dot{\div}A$. Since $S\dot{\div}A$ is closed under Cn, then $A\in S\dot{\div}A$, contrary to *success* since $A\in S$. From this contradiction we can conclude that $C\notin S\dot{\div}A$. It follows by (C'9) that $S\dot{\div}A\subseteq S\dot{\div}C$. Then we have $[S\dot{\div}C]\subseteq[S\dot{\div}A]$, and so $w\in[S\dot{\div}A]$. This means $\bigcup\{[S\div C]: [C]\subseteq[A]\}\subseteq[S\dot{\div}A]$. For the converse, it is clear for the particular cases that $A\in Cn_{S5}(\varnothing)$

or $A \notin S$. For the former, we know $X_A = X_A' = [S]$. For the latter, $X_A' = [S]$ and $[S \div A] = [S] \subseteq X_A$, so we have $X'_A \subseteq X_A$. Now we are going to consider the main case that $A \notin Cn_{S5}(\varnothing)$ and $A \in S$. It follows by (Def $\dot{-}$ from $\div$) that $[S \dot{-} A] = [\{C: C \in S \div (A \wedge C)\}]$. So we need to show that $[\{C: C \in S \div (A \wedge C)\}] \subseteq \bigcup\{[S \div C]: [C] \subseteq [A]\}$. It is equivalent to show that $\bigcap\{S \div C: [C] \subseteq [A]\} \subseteq \{C: C \in S \div (A \wedge C)\}$. Suppose for reductio that there is a formula D such that $D \in S \div C$ for any $[C] \subseteq [A]$ but $D \notin \{C: C \in S \div (A \wedge C)\}$. It follows that $D \notin S \div (A \wedge D)$. But we know $[A \wedge D] \subseteq [A]$. It follows from $D \in S \div C$ for any $[C] \subseteq [A]$ that $D \in S \div (A \wedge D)$, contradicting to $D \notin S \div (A \wedge D)$. Hence from this contradiction we obtain $\bigcap\{S \div C: [C] \subseteq [A]\} \subseteq \{C: C \in S \div (A \wedge C)\}$, and so $[S \dot{-} A] \subseteq \bigcup\{[S \div C]: [C] \subseteq [A]\}$. ⊣

Theorem 6: We only need to check $\boldsymbol{S} = \{X_A: A \in L\} \cup \{W\} = \{[S \dot{-} A]: A \in L\} \cup \{W\}$ since (Def $\boldsymbol{S}$ from $\div$) and (Def $\boldsymbol{S}$ from $\dot{-}$) are equivalent. First we have $[S] \subseteq [S \dot{-} A]$ for every A in L since $S \dot{-} A \subseteq S$ by *inclusion*. This means [S] is the innermost (minimal) sphere of $\boldsymbol{S}$. And it is clear that $[S \dot{-} A] \subseteq W$ for every A in L since $S \dot{-} A$ is consistent. It means W is the outermost (maximal) sphere of $\boldsymbol{S}$. Therefore $\boldsymbol{S}2$ and $\boldsymbol{S}3$ are satisfied. Now we check $\boldsymbol{S}$ satisfies $\boldsymbol{S}1$. It suffices to prove that for every A and C in L, $S \dot{-} A \subseteq S \dot{-} C$ or $S \dot{-} C \subseteq S \dot{-} A$ holds. This proposition has been proved in proposition 27. The last one is to check if it satisfies $\boldsymbol{S}4$. Let $\nvdash_{S5} \neg A$. We need to show there is a sphere $U \in \boldsymbol{S}$ such that $U \cap [A] \neq \varnothing$ and if $V \cap [A] \neq \varnothing$ implies $U \subseteq V$ for all $V \in \boldsymbol{S}$. It can be shown that $U = [S \dot{-} \neg A]$ satisfies this condition. Since $\nvdash_{S5} \neg A$, then by *success* of $\dot{-}$ we have $\neg A \notin S \dot{-} \neg A$. So it is clear that $[S \dot{-} \neg A] \cap [A] \neq \varnothing$. Now suppose for the contrary that there is some $V \in \boldsymbol{S}$ such that $V \cap [A] \neq \varnothing$ and $U \nsubseteq V$ (i.e., $V \subset U$ by $\boldsymbol{S}1$ which has been shown above to be hold). That is, by (Def $\boldsymbol{S}$ from $\dot{-}$), there is some $C \in L$ such that $[S \dot{-} C] \cap [A] \neq \varnothing$ and $[S \dot{-} C] \subset [S \dot{-} \neg A]$. Since $[S \dot{-} C] \cap [A] \neq \varnothing$, then we have $\neg A \notin S \dot{-} C$. It follows by (C'9) that $S \dot{-} C \subseteq S \dot{-} \neg A$ or, in other words, $[S \dot{-} \neg A] \subseteq [S \dot{-} C]$ contradicting the above. Hence we have $\boldsymbol{S}$ satisfies $\boldsymbol{S}4$. ⊣

Proposition 37: Assume S is consistent and A *PI-consistent*. If $S \cup \{A\}$ is consistent, then $S' = ((S \dot{-} \neg \mathbf{B}A) \cap \mathbf{B}S) + A = S$ is obviously consistent. Consider now $S \cup \{A\}$ is inconsistent. This means $\neg A$ and so $\neg \mathbf{B}A$ are in S. We know that $\mathbf{B}\neg A \vdash_{S5} \neg A \vdash_{S5} \neg \mathbf{B}A$ but the converse may not hold. Then by E2 we have $\mathbf{B}\neg A \leq \neg A$ and $\neg A \leq \neg \mathbf{B}A$. By E1, it follows that $\mathbf{B}\neg A \leq \neg \mathbf{B}A$. If we only give up $\neg A$ by the definition of contraction from epistemic entrenchment relation, then $\neg \mathbf{B}A$ may not be given up. However, if we give up $\neg \mathbf{B}A$ in the same way, then $\neg A$ and $\mathbf{B}\neg A$ can be guaranteed to be given up too. So consider contracting $\neg \mathbf{B}A$. Suppose for reductio that $((S \dot{-} \neg \mathbf{B}A) \cap \mathbf{B}S) + A$ is not consistent. Then we can find a finite minimal subset Σ of $(S \dot{-} \neg \mathbf{B}A) \cap \mathbf{B}S$ such that $\Sigma \vdash_N \neg \mathbf{B}A$ by *compactness* of Cm. It follows by proposition 23 that $\Sigma \vdash_{S5} \neg \mathbf{B}A$ since Σ consists all belief atoms. And we know $\neg \mathbf{B}A$ will be given up by the definition of $\dot{-}$ from $\leq$, then $\wedge\Sigma$ should be discarded too for it is in S. By E3, we can find a C_i $(1 \leq i \leq n)$ such that $C_i \leq \wedge\Sigma$. It shows that at least one formula C_i of Σ will not appear in the new set after the contraction on S by $\neg \mathbf{B}A$. Since Σ is minimal, then $\Sigma \setminus \{C_i\} \nvdash_{S5} \neg \mathbf{B}A$. And all those Σs will be discarded with some formulae so that each of them cannot deduce $\neg \mathbf{B}A$ in consequence relation $\vdash_{S5}$. Neither can it be deduced under $\vdash_N$ from $\Sigma \setminus \{C_i\}$ since $\Sigma \setminus \{C_i\}$ consists all and only belief atoms. This means $(S \dot{-} \neg \mathbf{B}A) \cap \mathbf{B}S$ cannot deduce $\neg \mathbf{B}A$ under the consequence relation $\vdash_N$. So $S' = ((S \dot{-} \neg \mathbf{B}A) \cap \mathbf{B}S) + A$ is *PI-consistent* and then consistent. ⊣

Proposition 39: We need to show that $A \leq C$ if and only if $A \leq' C$. Suppose $A \leq C$ holds. Then by definition 30 we have $A \notin S \dot{-} C$ or $\vdash_{S5} C$. If $\vdash_{S5} C$ holds, then $Cn_{S5}(A) = Cn_{S5}(A \wedge C)$. It follows by *extensionality* of $\div$ that $S \div (A \wedge C) = S \div A$. There are two sub-cases in this case: If A is also an **S5** theorem, then we have $\vdash_{S5} (A \wedge C)$. It suffices to show that $A \leq' C$. If A is not an **S5** theorem, then by success of $\div$ we have $A \notin S \div A$, i.e., $A \notin S \div (A \wedge C)$. This also shows that $A \leq' C$. Now consider the main case that $A \notin S \dot{-} C$. It follows by (Def $\dot{-}$ from $\div$) that $A \notin \{D: D \in S \div' (C \wedge D)\}$. And by observation that $\div$' is equivalent to $\div$, so we have $A \notin \{D: D \in S \div (C \wedge D)\}$. This means $A \notin S \div (C \wedge A)$, i.e., $A \leq' C$, as required.

For the converse part, assume $A \leq' C$. Then by Definition 31 we have $A \notin S \div (A \wedge C)$ or $\vdash_{\mathbf{S5}}(A \wedge C)$. It is trivial to obtain $A \leq C$ in the $\vdash_{\mathbf{S5}}(A \wedge C)$ case. We only consider the main case that $A \notin S \div (A \wedge C)$. It follows by (Def $\div$ from $\dotdiv$) that $A \notin S \cap Cn_{\mathbf{S5}}(S \dotdiv' (A \wedge C) \cup \{\neg(A \wedge C)\})$. And we know $\dotdiv$' is equivalent to $\dotdiv$ by observation, so $A \notin S \cap Cn_{\mathbf{S5}}(S \dotdiv (A \wedge C) \cup \{\neg(A \wedge C)\})$. If $A \notin S$ then $A \notin S \dotdiv C$ by *inclusion* of $\dotdiv$. This means $A \leq C$. If $A \in S$ then $A \notin Cn_{\mathbf{S5}}(S \dotdiv (A \wedge C) \cup \{\neg(A \wedge C)\})$. Suppose for reductio that $A \in S \dotdiv C$. And we have $S \dotdiv C \subseteq S \dotdiv (A \wedge C)$ by (C'7) since C is not an **S5** theorem. Then $A \in S \dotdiv (A \wedge C)$ and $A \in Cn_{\mathbf{S5}}(S \dotdiv (A \wedge C) \cup \{\neg(A \wedge C)\})$, a contradiction. From this contradiction we obtain $A \notin S \dotdiv C$, i.e. $A \notin S \dotdiv C$. ┤

Theorem 8: We only consider the relation obtained by (Def $\leq$ from $\dotdiv$) since they are equivalent by proposition 39.

First we check E1. Assume $A \leq C$ and $C \leq D$, we need to show $A \leq D$. It follows by (Def $\leq$ from $\dotdiv$) that $A \notin S \dotdiv C$ or $\vdash_{\mathbf{S5}} C$ and $C \notin S \dotdiv D$ or $\vdash_{\mathbf{S5}} D$. If $\vdash_{\mathbf{S5}} D$ holds then we have $A \leq D$, as required. Suppose that $\nvdash_{\mathbf{S5}} D$. Then we have $C \notin S \dotdiv D$. It is impossible that $\vdash_{\mathbf{S5}} C$ since $C \notin S \dotdiv D$ and $S \dotdiv D$ is closed under $Cn_{\mathbf{S5}}$, and so $A \notin S \dotdiv C$. It follows by (C'9) and $C \notin S \dotdiv D$ that $S \dotdiv D \subseteq S \dotdiv C$. Hence we obtain $A \notin S \dotdiv D$ by $A \notin S \dotdiv C$ and $S \dotdiv D \subseteq S \dotdiv C$. This shows that $A \leq D$.

Next we check E2. Assume that $A \vdash_{\mathbf{S5}} C$, we need to show $A \leq C$. Suppose for reductio that $A \nleq C$, that is, $A \in S \dotdiv C$ and $\nvdash_{\mathbf{S5}} C$ by (Def $\leq$ from $\dotdiv$). It follows that $C \in S \dotdiv C$ since $A \vdash_{\mathbf{S5}} C$ and $S \dotdiv C$ is closed under $Cn_{\mathbf{S5}}$. This contradicts *success* of $\dotdiv$ since $\nvdash_{\mathbf{S5}} C$. Hence we obtain that $A \leq C$.

Now we check E3. Suppose for reductio that $A \nleq A \wedge C$ and $C \nleq A \wedge C$. It follows by (Def $\leq$ from $\dotdiv$) that $A \in S \dotdiv (A \wedge C)$ and $\nvdash_{\mathbf{S5}} A \wedge C$, $C \in S \dotdiv (A \wedge C)$ and $\nvdash_{\mathbf{S5}} A \wedge C$. This means $A \wedge C \in S \dotdiv (A \wedge C)$, contradicting to *success* of $\dotdiv$ since $\nvdash_{\mathbf{S5}} A \wedge C$. Hence we have $A \leq A \wedge C$ or $C \leq A \wedge C$.

Next we check E4. Suppose $A \notin S$, we need to show that for all C in L such that $A \leq C$. It is clear that $A \notin S \dotdiv C$ for any C in L. So we have $A \leq C$ by (Def $\leq$ from $\dotdiv$).

Last in this proof we check E5. Suppose for all C in L that $C \leq A$. It follows by (Def $\leq$ from $\dotdiv$) that $C \notin S \dotdiv A$ or $\vdash_{\mathbf{S5}} A$. And suppose for reductio that $\nvdash_{\mathbf{S5}} A$. Then we have $C \notin S \dotdiv A$ for any C in L. If C is an S5 theorem then it must be that $C \in S \dotdiv A$, a contradiction. Hence we can conclude that $\vdash_{\mathbf{S5}} A$ from this contradiction, as required. ┤

Proposition 42: If $A \vdash C$ but $C \nvdash A$, then by principle 2 in definition 34 we have $A < C$. If $C \vdash A$ but $A \nvdash C$, then we can obtain $C < A$ for the same reason. If $A \nvdash C$ and $C \nvdash A$, then by principle 3 we have $A < A \wedge C$ or $C < A \wedge C$. Suppose $A < A \wedge C$ holds. We have $A \wedge C < C$ since $A \wedge C \vdash C$ and $C \nvdash A \wedge C$. So it follows by principle 1 that $A < C$. For the other case, suppose that $C < A \wedge C$ holds. It can be derived exactly the same way that $C < A$. ┤

Theorem 9: It is easy to check that # satisfies *consistency*. We know if A is *PI-consistent*, then S*A is *PI-consistent*. There is at least one consistent stable set T containing S*A and f selects the stable set with smallest information value.

Now we check # satisfies *success*. It follows directly since f(Z) contains S'=S*A and $A \in S*A$ (by *success* of *).

And it is trivial to show # satisfies *stability* because f(Z) is a stable set.

For *vacuity*, suppose that $\neg \mathbf{B} A \notin S$, we need to show f(Z)=S. By *vacuity* of *, we have S*A=S. So there is only one stable set S containing S since every two different stable sets cannot contain each other. And so f({S})=S.

For *extensionality*, suppose Cm(A)=Cm(C). By *extensionality* of *, we have S*A=S*C and then Exp(S*A)=Exp(S*C). Since f is a function, it follows that f(Exp(S*A))=f(Exp(S*C)), that is, S#A=S#C. ┤

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