A New Kind of Rough Implication

XUE Zhan-ao[1、2] HE Hua-can[1]

(1.*Department of Computer Science and Engineering, Northwestern Polytechnical University, Xi'an 710072. P.R..China*； 2 *Department of Computer Science, Henan Normal University, Xinxiang 453002. P.R.. China*)

Abstract： The new definition of rough set and rough implication operator are investigated by Stone algebra to improve the shortage of [4] and new rough operators such as rough intersection, rough union, rough complement are introduced. Furthermore we study the relations of the proposed operations and their characteristics, and also point out that the proposed rough implication operator is consistent with that of three-valued Lukasiewicz logic and has many good properties.

Keywords： Rough Logic, Stone Algebra, Rough Implication, Approximation Spaces

CLC Number: TP18

1. Introduction

Rough set theory, introduced by Zdzislaw Pawlak in the early 1980s[1、2、3], is a new mathematical tool to deal with many problems such as vagueness, uncertainty, incomplete data and reasoning. Now there are lots of papers about rough logic thanks to its abroad application, but all the rough implication operators exist some defects for instance, $mng(j \to y) = \langle -B \cup C \cup (D \cap -A), -B \cup D \rangle$ (Let $\underline{A} = A, \overline{A} = B, \underline{B} = C, \overline{B} = D$) in [4] cannot imply $B^c \to A^c \approx A \to B$. To improve those defects we apply Stone algebra to rough set algebra system and give a new rough implication operator with good properties that is consistent with that of three-valued Lukasiewicz logic.

2. Rough set theory

Definition2.1 Let *U* be the universe set and *R* be an equivalent relation on *U.* A pair (*U, R*) is called an approximation space. If $X \subseteq U$ is an arbitrary set, then two approximations are formally defined as follows:

$$\underline{X} = \{x \mid x \in U, [x]_R \subseteq X\}, \ \overline{X} = \{x \mid x \in U, [x]_R \cap X \neq f\}.$$

Where $[x]_R$ is an equivalent class containing x. $\underline{X}$ is called lower approximation of *X*, $\overline{X}$ is called upper approximation of *X*. The approximated set *X* lies between its lower and upper approximations:

$$\underline{X} \subseteq X \subseteq \overline{X}.$$

We have

$$-\overline{X} \subseteq -X \subseteq -\underline{X}$$

with $Z \subseteq U$ and $-Z$ is the complement of Z in U.

For each $X \subseteq U$, a rough set is a pair $\langle \underline{X}, \overline{X} \rangle$. We denote the empty set f by $\langle \underline{f}, \overline{f} \rangle \approx \langle f, f \rangle$, the universe set U by $\langle \underline{U}, \overline{U} \rangle \approx \langle U, U \rangle$ and the power set of U by $\Re(U)$.

With every pair of approximations, we can distinguish three distinct regions on *U*:

$POS_R(X) = \underline{X}$; R-positive region of *X*,

$BND_R(X) = \overline{X} - \underline{X}$; R-boundary region of *X*,

$NEG_R(X) = U - \overline{X}$: R-negative region of *X*.

The positive region of *X* is the set of all objects which can be certainly classified as elements of *X*. The negative region of *X* is the set of all objects which can not be certainly classified as elements of *X* and the boundary region contains all objects which can be classified as elements of *X*. Hence, imprecision in rough sets is due to the boundary region. Obviously, crisp sets have no boundary region.

Definition 2.2 Let $A, B \in \Re(U)$, the inclusion relation of two rough sets is defined as,

$A \tilde{\subset} B$ if and only if $\overline{A} \subseteq \overline{B}$ **and** $\underline{A} \subseteq \underline{B}$;

The equivalent relation of two rough sets is defined as,

$A \approx B$ if and only if $\overline{A} = \overline{B}$ and $\underline{A} = \underline{B}$.

Definition 2.3 Let $A, B \in \Re(U)$, the intersection of two rough sets is a rough set in approximation space and is defined as,

$A \cap B \approx \langle \underline{A} \cap \underline{B}, \overline{A} \cap \overline{B} \rangle$,

The union of two rough sets is a rough set in approximation space and is defined as,

$A \cup B \approx \langle \underline{A} \cup \underline{B}, \overline{A} \cup \overline{B} \rangle$,

The complement of *A* is a rough set in approximation space and is defined as,

$A^c \approx \langle -\overline{A}, -\underline{A} \rangle$,

The pseudocomplement of *A* is a rough set in approximation space and is defined as,

$A^* \approx \langle -\underline{A}, -\underline{A} \rangle$,

where $X \subseteq U$, $-X$ is the complement of *X* in U .

Theorem 2.1 Suppose $A, B \in \Re(U)$, then

$$\underline{A \cap B} = \underline{A} \cap \underline{B}, \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B};$$

$$\underline{A \cup B} \supseteq \underline{A} \cup \underline{B}, \qquad \overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Proof. Theorem 2.1 follows from [1]~[3] and [7].

Theorem 2.2 Suppose A^c is the complement of A in U, A^* is the pseudocomplement of A in U, then

(1) $A^c \subseteq A^*$; (2) $A^{**} \subseteq A^{c*}$;

(3) $A^c \cup A^* = A^*$, $A^c \cap A^* = A^c$;

(4) $A^{c*c} = A^{c**} = \langle -\overline{A}, -\overline{A} \rangle$;

(5) $A^{cc*} = A^{*c*} = A^{***} = A^{**c} = A^{*cc} = A^*$;

(6) $A^{ccc} = A^c$; (7) $A^{c*c*} = A^{c*}$.

Proof. Theorem 2.2 can be proved easily from Definition 2.3.

Remark. Obviously, (4) is the pseudocomplement A*(i.e. $\langle -\overline{A}, -\overline{A} \rangle$) in [4], and the pseudocomplement defined in this paper is the dual pseudocomplement $\langle -\underline{A}, -\underline{A} \rangle$ in [4]. Also the pseudocomplement and dual pseudocomplement in [4] cannot imply $A^c \approx \langle -\overline{A}, -\underline{A} \rangle$ and $B^c \to A^c \approx A \to B$, while the complement and pseudocomplement in this paper can imply dual pseudocomplement, and the complement and dual pseudocomplement can imply pseudocomplement.

Theorem 2.3 Suppose $A, B \in \Re(U)$, then,

$$(A \cap B)^c \approx A^c \cup B^c; \qquad (A \cup B)^c \approx A^c \cap B^c;$$

$$(A \cap B)^* \approx A^* \cup B^*; \qquad (A \cup B)^* \approx A^* \cap B^*$$

Proof. Theorem 2.3 is easy to be proved by Definition 2.3.

3. New implication operator

We assume familiarity with the basic concepts of lattice theory, universal algebra, and logic. For definitions not explained here we refer the reader to [12] for lattice theory and universal algebra, and to [13] for logic. In this paper we directly use the Stone algebra of rough sets that has been proved in [9], and we propose the new implication operator to improve the results in [4].

Let B be a Boolean algebra, F be a filter on B, and

$$\langle B, F \rangle = \{\langle a, b \rangle \mid a, b \in B, a \le b, a + (-b) \in F\}.$$

If $B = F$, then we usually write B for $\langle B, F \rangle$ and now we define the following operations on $\langle B, F \rangle$:

$\langle a,b \rangle + \langle c,d \rangle = \langle a+c, b+d \rangle$; $\langle a,b \rangle \cdot \langle c,d \rangle = \langle a \cdot c, b \cdot d \rangle$;

$\langle a,b \rangle^c = \langle -b,-a \rangle$; $\langle a,b \rangle^* = \langle -a,-a \rangle$.

A model of b is a pair $\langle W, v \rangle$,where W is a set, and $v : p \to \Re(w) \times \Re(w)$, a mapping -called the *valuation function* with every $p \in P$.

If $v(p) = \langle A, B \rangle$, then $A \subseteq B$.

The equality $v(p) = \langle A, B \rangle$ means that

***P* holds at all states of *A*, and does not hold at any state outside *B* .**

The following characterization of *valuation* demonstrates the relationship to three-valued Lukasiewicz logic: For each $p \in P$ let $v_p : w \to 3 = \{0, \frac{1}{2}, 1\}$ be a mapping. Then $v : P \to \Re(W)$ defined by

$v(P) = \langle \{w \in W, v_p(w) = 1\}, \{w \in W, v_p(w) \neq 0\} \rangle$ is a valuation.

Conversely, if $w \in A$, then $v_p(w) = 1$; if $w \in B - A$,then $v_p(w) = \frac{1}{2}$; otherwise, $v_p(w) = 0$.

Given a model $\Re = \langle W, v \rangle$, we define its meaning function **mng:** $Fml \to \Re(w) \times \Re(w)$ as an extension of the valuation v as follows:

$mng(\to p) = \langle w, w \rangle$.

For each $\forall p \in P$, $mng(p) = v(P)$.

If $mng(j) = \langle A, B \rangle$ and $mng(y) = \langle C, D \rangle$ then

$mng(j \wedge y) = \langle A \cap C, B \cap D \rangle$; $mng(j \vee y) = \langle A \cup C, B \cup D \rangle$

$mng(j^c) = \langle -B, -A \rangle$; $mng(j^*) = \langle -A, -A \rangle$;

$mng(0) = \langle f, f \rangle$; $mng(1) = \langle U, U \rangle$.

Here, -*A* is the complement of *A* in $\Re(W)$.

Let $ran(mng) = \{mng(j) : j \in Fml\}$. We define operations on ran(***mng***) in the obvious way. It is easy to prove following equalities from above definition

$$mng(j \wedge y) = mng(j) \cdot mng(y) \text{ ; } \quad mng(j \vee y) = mng(j) + mng(y) \text{ ;}$$

$$mng(j^{c}) = mng(j)^{c}; \qquad mng(j^{*}) = mng(j)^{*}.$$

We define additional operations on ***Fml*** by

$$j \rightarrow y = j^{c} \vee y \vee (j^{*} \wedge y^{c*});$$

$$j \leftrightarrow y = (j \rightarrow y) \wedge (y \rightarrow j);$$

$$mng(j \rightarrow y) = mng(j) \rightarrow mng(y). \qquad (\text{i})$$

Hence, we have the following equality

$$mng(j \rightarrow y) = mng(j^{c}) + mng(y) + mng(j^{*}) \cdot mng(y^{c*})$$

$$= mng(j)^{c} + mng(y) + mng(j)^{*} \cdot mng(y)^{c*}$$

$$= \langle -B, -A \rangle + \langle C, D \rangle + \langle -A, -A \rangle \cdot \langle D, D \rangle$$

$$= \langle -B \cup C \cup (D \cap -A), -A \cup D \rangle \qquad (\text{ii})$$

Theorem 3.1[4] If $j, y \in Fml$,then

(1) $\Re \models j \leftrightarrow y$ if and only if $mng(j) = mng(y)$.

(2) $\Re \models \rightarrow p \leftrightarrow j$ if and only if $\Re \models j$ **.**

(3) if h is a *homomorphism* of **Fml** and $\Re = \langle W, v \rangle$,then there exists some $\Re$ such that

$mng_{\Re} = mng_{\Re} \circ h$.

Proof See Theorem 5 in [4].

4. Properties of implication operator

In this section, we define a new implication operator $\rightarrow$**,** and investigate its properties in rough logic system.

Let $mng(j) = \langle \underline{A}, \overline{A} \rangle$, $mng(y) = \langle \underline{B}, \overline{B} \rangle$, $mng(b) = \langle \underline{C}, \overline{C} \rangle$ and by the knowledge in Section 3, we have

$$(j \vee y)^{c} = j^{c} \wedge y^{c}; \qquad (j \wedge y)^{c} = j^{c} \vee y^{c};$$

$$(j \vee y)^{*} = j^{*} \wedge y^{*}; \qquad (j \wedge y)^{*} = j^{*} \vee y^{*}$$

$$j^{c*c} = j^{c**}; \qquad j^{ccc} = j^{c}; \qquad j^{c*c*} = j^{c*}$$

$$j^{cc*} = j^{*c*} = j^{***} = j^{**c} = j^{*cc} = j^{*};$$

$$mng(j^{c*c}) = \langle -\overline{A}, -\overline{A} \rangle; \quad mng(j^{c}) = \langle -\overline{A}, -\underline{A} \rangle;$$

$$mng(j^{c}) = \langle -\underline{A}, -\underline{A} \rangle; \qquad mng(j^{c*}) = \langle \overline{A}, \overline{A} \rangle .$$

$$mng(j \to y) = mng(j^{c} \vee y \vee (j^{*} \wedge y^{c*}))$$

$$= \langle -\overline{A} \cup \underline{B} \cup (\overline{B} \cap -\underline{A}), -\underline{A} \cup \overline{B} \rangle \qquad \text{(iii)}$$

Theorem 4.1 Let $A, B \in \Re(U)$, then the following two equalities are equivalent:

(1) $A \to B = \langle -\overline{A} \cup \underline{B} \cup (\overline{B} \cap -\underline{A}), -\underline{A} \cup \overline{B} \rangle$ **;**

(2) $A \to B = \langle (-\underline{A} \cup \underline{B}) \cap (-\overline{A} \cup \overline{B}), -\underline{A} \cup \overline{B} \rangle$ **.**

Proof. Theorem 4.1 is easy to be proved from (**iii**).

Theorem 4.2 Let $A, B \in \Re(U)$, then

(1) $U \to f \approx f$ **;**

(2) $f \to U \approx U$;

(3) $f \to A \approx U$;

(4) $U \to A \approx A$;

(5) $A \to A \approx U$ **;**

(6) $A \to (B \to C) \approx B \to (A \to C)$;

(7) if $A \tilde{\subset} B$ **, then** $B \to C \tilde{\subset} A \to C$;

(8) if $B \tilde{\subset} C$ **, then** $A \to B \tilde{\subset} A \to C$ **;**

(9) $B \tilde{\subset} A \to B$;

(10) $A \to B \approx U$ if and only if $A \tilde{\subset} B$;

(11) $B^{c} \to A^{c} \approx A \to B$.

(12) $A \to f = A^{c}$

Proof. Here we only prove (6),(7),(11) and others can be proved easily by Theorem 4.1,(**i**) and (**iii**).

Proof of (6) Since $j \to (y \to b) = j^{c} \vee (y \to b) \vee (j^{*} \wedge (y \to b)^{c*})$

$$= j^{c} \vee (y^{c} \vee b \vee (y^{*} \wedge b^{c*})) \vee (j^{*} \wedge (y^{c} \vee b \vee (y^{*} \wedge b^{c*}))^{c*})$$

$$= j^{c} \vee y^{c} \vee b \vee (y^{*} \wedge b^{c*}) \vee (j^{*} \wedge (y^{*} \vee b^{c*} \vee (y^{*} \wedge b^{c*})))$$

$$= j^{c} \vee y^{c} \vee b \vee (y^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{*}) \vee (j^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{*} \wedge b^{c*})$$

we get $y \to (j \to b) = y^{c} \vee (j \to b) \vee (y^{*} \wedge (j \to b)^{c*})$

$$= y^{c} \vee (j^{c} \vee b \vee (j^{*} \wedge b^{c*})) \vee (y^{*} \wedge (j^{c} \vee b \vee (j^{*} \wedge b^{c*}))^{c*})$$

$$= y^{c} \vee j^{c} \vee b \vee (j^{*} \wedge b^{c*}) \vee (y^{*} \wedge (j^{*} \vee b^{c*} \vee (j^{*} \wedge b^{c*})))$$

$$= y^{c} \vee j^{c} \vee b \vee (j^{*} \wedge b^{c*}) \vee (y^{*} \wedge j^{*}) \vee (y^{*} \wedge b^{c*}) \vee (y^{*} \wedge j^{*} \wedge b^{c*})$$

$$= j^{c} \vee y^{c} \vee b \vee (y^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{*}) \vee (j^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{*} \wedge b^{c*})$$

Combining (i) we can obtain $A \to (B \to C) \approx B \to (A \to C)$ and this completes the proof of (6).

Proof of (7) Since by (**iii**),we have

$$B \to C = \langle (-\underline{B} \cup \underline{C}) \cap (-\overline{B} \cup \overline{C}), -\underline{B} \cup \overline{C} \rangle$$

$$A \to C = \langle (-\underline{A} \cup \underline{C}) \cap (-\overline{A} \cup \overline{C}), -\underline{A} \cup \overline{C} \rangle$$

Again since $A \tilde{\subset} B$ iff $\underline{A} \subseteq \underline{B}, \overline{A} \subseteq \overline{B}$ iff $-\underline{B} \subseteq -\underline{A}, -\overline{B} \subseteq -\overline{A}$,

then $-\underline{B} \cup \underline{C} \subseteq -\underline{A} \cup \underline{C}, -\overline{B} \cup \overline{C} \subseteq -\overline{A} \subseteq \overline{C}, -\underline{B} \cup \overline{C} \subseteq -\underline{A} \cup \overline{C}$

Hence, $B \to C \tilde{\subset} A \to C$.

Proof of (11)

$$j^{c} \to y^{c} = j^{cc} \vee y^{c} \vee (j^{c*} \wedge y^{cc*}) = y^{c} \vee j \vee (y^{*} \wedge j^{c*})$$

$$y \to j = y^{c} \vee j \vee (y^{*} \wedge j^{c*})$$

Hence, $B^{c} \to A^{c} \approx A \to B$.

Remark Theorem4.2 shows that the new implication operator satisfies the basic properties ((1)~(5)), monotonicity((7),(8)) of implication, and (6) and (9) yield the deductibility and preserve-order respectively, also (11) and (12) give the equivalence of inversely negative proposition and original proposition.

Theorem 4.3 Let $A, B, C \in \Re(U)$, then

(1) $A \to (B \cap C) \approx (A \to B) \cap (A \to C)$;

(2) $(A \cup B) \to C \approx (A \to C) \cap (B \to C)$;

(3) $A \to (B \cup C) \approx (A \to B) \cup (A \to C)$;

(4) $(A \cap B) \to C \approx (A \to C) \cup (B \to C)$.

Proof. Here we only prove (2),(4), and others can be proved easily.

Proof of (2)

$$(j \to b) \wedge (y \to b) = (j^{c} \vee b \vee (j^{*} \wedge b^{c*})) \wedge (y^{c} \vee b \vee (y^{*} \wedge b^{c*}))$$

$$= (j^{c} \wedge y^{c}) \vee (j^{c} \wedge b) \vee (j^{c} \wedge y^{*} \wedge b^{c*}) \vee (b \wedge y^{c}) \vee b \vee (b \wedge y^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{c} \wedge b^{c*}) \vee (b \wedge j^{*} \wedge b^{c*}) \vee (j^{*} \wedge y^{*} \wedge b^{c*})$$

$$= (j^{c} \wedge y^{c}) \vee b \vee (j^{*} \wedge y^{*} \wedge b^{c*})$$

$$(j \vee y) \to b = (j \vee y)^{c} \vee b \vee ((j \vee y)^{*} \wedge b^{c*})$$

$$= (j^{c} \wedge y^{c}) \vee b \vee (j^{*} \wedge y^{*} \wedge b^{c*})$$

Hence, $(A \cup B) \to C \approx (A \to C) \cap (B \to C)$

Proof of (4)

$$(j \wedge y) \to b = (j \wedge y)^{c} \vee b \vee ((j \wedge y)^{*} \wedge b^{c*})$$

$$= (j^{c} \vee y^{c}) \vee b \vee ((j^{*} \vee y^{*}) \wedge b^{c*})$$

$$= (j^{c} \vee y^{c}) \vee b \vee (j^{*} \wedge b^{c*}) \vee (y^{*} \wedge b^{c*})$$

$$(j \to b) \vee (y \to b) = (j^{c} \vee b \vee (j^{*} \wedge b^{c*})) \vee (y^{c} \vee b \vee (y^{*} \wedge b^{c*}))$$

$$= (j^{c} \vee y^{c}) \vee b \vee (j^{*} \wedge b^{c*}) \vee (y^{*} \wedge b^{c*})$$

Hence, $(A \cap B) \to C \approx (A \to C) \cup (B \to C)$.

Theorem 4.4 Let $A, B \in \Re(U)$, then

(1) $A \,\tilde{\subset}\, ((A \to B) \to B)$;

(2) $(A \to B) \approx \cup\{X \mid A \,\tilde{\subset}\, X \to B\}$.

Proof. It is easily proved by (6) and (10) of Theorem 4.2.

Remark Theorem 4.4 and (6) and (9) of Theorem 4.2 support the reasoning process of $A', A \to B \Rightarrow B'$.

5. Conclusion

Rough implication operator is the emphasis and difficulty in the study of rough logic and the rough implication operation in [4] can not imply $B^{c} \to A^{c} \approx A \to B$, i.e. the inversely negative proposition and original proposition are not equivalent. Due to the shortages of rough pseudocomplement （*） and dual pseudocomplement （$^-$） in [4], in this paper with the view of Stone algebra we begin with rough set operation to redefine the rough intersection, rough union, rough complement and rough implication operator whose relations and properties have been investigated. It is shown that the proposed implication operator is consistent with that of three-valued Lukasiewicz logic and has many good properties.

References

1. Pawlak Z. Rough Sets. International Journal of Computer and Information Sciences, 11(1982):341~356.

2. Pawlak Z. Rough Sets-Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers. Dordrecht, 1991.

3. Pawlak Z. Vagueness and Uncertainty: A Rough Set Perspective, Computational Intelligence.11 (1995):227~232.

4. Duntsch I. A logic for Rough Sets. Theoretical Computer Siences, 179 (1997):427~436.

5. E.Orlowska Reasoning about Vague Concepts. Bull. Pol. Ac: Mathematics, 35(1987):643~652.

6. Donald W.Barnes and John M. Mack. An Algebraic Introduction to Mathematical Logic. Spring-Verlag, New York, 1975.

7. Zhang Wenxiu etc.. Rough set theory & method. Science Press, Beijing, 2001.7 (in Chinese).

8. Zhu F. He H C. The Axiomatization of the Rough Set. Journal of Computer (in Chinese). 2000, Vol.23, No.3:330~333.

9. Pomykala J. and Pomykala J.A. The Stone Algebra of Rough Sets. Bull. Polish Acad. Sci. Math. 36(1988):495~508.

10. Chang C C. Algebraic analysis of many-valued [J]. Trans.Amer.Math.Soc.1958, 88:467~490.

11. Wang G J. MV-Algebras, BL-Algebras, R_0-Algebras, and Multiple-Valued Logic. Fuzzy Systems and Mathematics (in Chinese).2002, 16(2):1~15.

12.Balbes R. and Dwinger P. Distributive Lattices (University of Missouri Press, Columbia,1974).

13.Andreka H. and Nemeti I. General algebraic logic: a perspective on 'What is logic', in: Gabbay D.M. ,ed., What is a Logical System? Studies in Logic and Computation, Vol. 4 (Clarendon Press, Oxford,1994) 393~443.